LIE BIALGEBROIDS AND POISSON GROUPOIDS

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1. Introduction

Lie bialgebras arise as infinitesimal invariants of Poisson Lie groups. A Lie bialgebra is a Lie algebra $\mathfrak{g}$ with a Lie algebra structure on the dual $\mathfrak{g}^*$ which is compatible with the Lie algebra $\mathfrak{g}$ in a certain sense. For a Poisson group G, the multiplicative Poisson structure π induces a Lie algebra structure on the Lie algebra dual $\mathfrak{g}^*$ which makes $(\mathfrak{g}, \mathfrak{g}^*)$ into a Lie bialgebra. In fact, there is a one-one correspondence between Poisson Lie groups and Lie bialgebras if the Lie groups are assumed to be simply connected [7], [16], [19]. The importance of Poisson Lie groups themselves arises in part from their role as classical limits of quantum groups [8] and in part because they provide a class of Poisson structures for which the realization problem is tractable [15].

Poisson groupoids were introduced by Weinstein [24] as a generalization of both Poisson Lie groups and the symplectic groupoids which arise in the integration of arbitrary Poisson manifolds [4], [11]. He noted that the Lie algebroid dual A^*G of a Poisson groupoid G itself has a Lie algebroid structure, but did not develop the infinitesimal structure further. In this paper we introduce and study a natural infinitesimal invariant for Poisson groupoids, the Lie bialgebroids of the title. Our ultimate purpose is to develop a Lie theory for Poisson groupoids parallel to that for Poisson groups. In this paper we are primarily concerned with the first half of this process; that is, with the construction of the Lie bialgebroid of a Poisson groupoid.

After the preliminary §2, in which we describe the generalization to arbitrary Lie algebroids of the exterior calculus and Schouten calculus, in §3 we define a Lie bialgebroid to be a Lie algebroid A whose dual A^* is also equipped with a Lie algebroid structure, such that the coboundary operator $d_*\colon A \longrightarrow \wedge^2(A)$ associated to A^* satisfies a cocycle equation with respect to $\Gamma(A)$, the Lie algebra of sections of A. This is clearly a straightforward extension of the concept of a Lie bialgebra [16] but cannot be formalized in Lie algebroid cohomological terms since there is no satisfactory adjoint representation for a general Lie algebroid. Most of §3 is devoted to proving that this definition is self-dual: if (A, A^*) is a Lie bialgebroid, then (A^*, A) is also. In §4, we briefly consider the special case of Lie bialgebroids satisfying a triangularity condition, which include some important examples such as the usual triangular Lie bialgebras and Lie bialgebroids associated to Poisson manifolds.

The techniques used in §§2–4 are similar to those known for Lie bialgebras. It would be possible, by suitably generalizing the proof for Poisson groups, to prove

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already at this stage that if G is a Poisson groupoid, then (AG, A^*G) is a Lie bialgebroid with respect to the canonical structures. However we prefer to deduce this in §8 from the results of §6 and §7.

The integrability of Lie bialgebras is established by noting that for a Poisson Lie group G, the Poisson tensor can be right-translated to the identity element and gives a Lie group 1-cocycle on G with values in $\mathfrak{g}^*$. Since every finite-dimensional Lie algebra is integrable to a simply-connected Lie group, the integrability problem for Lie bialgebras is equivalent to the problem of lifting Lie algebra cocycles to Lie group cocycles, and this is always possible when the Lie group is simply-connected.

In the case of a Lie bialgebroid (A, A^*), there is first of all the fact that not all Lie algebroids A integrate to Lie groupoids; in discussing the integrability of Lie bialgebroids we will however always assume that the Lie algebroids themselves are integrable. More importantly, what needs to be lifted is now a cocycle of an infinite-dimensional Lie algebra $\Gamma(A)$, and here there is no general theory to apply. Instead we propose to make use of the machinery developed for Lie algebroid morphisms in [9]. To explain how this works, we have to recall briefly the relation between Poisson Lie groups and Lie bialgebras from this viewpoint.

In [18, §4] one of us analysed the relations between a general Poisson Lie group and its Lie algebra dual in terms of groupoids and Lie algebroids. In particular, it is shown that a Lie group with a Poisson structure is a Poisson Lie group if and only if the Poisson bundle map from T^*G to TG is a Lie groupoid morphism, where T^*G is the canonical cotangent groupoid over the base $\mathfrak{g}^*$ and TG is the tangent group.

This result can be easily generalized to the case of groupoids in a straightforward manner (§8). This suggests that there should be an equivalent description of Lie bialgebroids in terms of Lie algebroid morphisms, and this is in fact the content of §6. Such a description appears to be more complicated than Definition 3.1, but should be more tractable when dealing with the integrability problem. We prove in Theorem 6.2 that a Lie algebroid A with a Lie algebroid structure on its dual A^* is a Lie bialgebroid if and only if the Poisson bundle map $\pi_A^{\#}\colon T^*(A) \longrightarrow TA$ is a Lie algebroid morphism with respect to the Lie algebroid structure on $T^*(A)$ with base A^*. This result, the proof of which extends over all of §6, is the main result of the paper, and is the key to the relationship between Lie bialgebroids and Poisson groupoids. In order to formulate and prove Theorem 6.2, we need to develop the basic properties of the tangent and cotangent bundles of a Lie algebroid, and we do this in §5. In particular, we show that the tangent $TA \longrightarrow TP$ of any Lie algebroid $A \longrightarrow P$ is itself a Lie algebroid in a natural way; this structure is dual to the tangent Poisson structure [6] for the dual Poisson structure on A^*. Similarly, before we can deal with Poisson groupoids we need to explicate the double structures associated with the tangent and cotangent bundles of a Lie groupoid, and we do this in §7. Sections 5 and 7 may be regarded as a direct continuation of some aspects of [18].

Finally, we note that given a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ it is a simple matter to verify that $\mathfrak{g} \oplus \mathfrak{g}^*$ has a Lie algebra structure for which $\mathfrak{g}$ and $\mathfrak{g}^*$ are subalgebras, and for which the cross terms are given by the two coadjoint representations. This is the so called double Lie algebra of $(\mathfrak{g}, \mathfrak{g}^*)$. The importance of the double Lie algebra lies in the fact that it embodies almost all the information of the Lie bialgebra, and its corresponding Lie group, the so called double Lie group, turns out to be a natural object describing the dressing transformations. For a Lie bialgebroid (A, A^*) on an arbitrary base, the usual formula does not give a Lie algebroid structure on $A \oplus A^*$. In fact, evidence suggests that the correct replacement should be $T^*(A)$. However, the structure on $T^*(A)$ which will play the role of the double Lie algebra is not clear, and we hope to explore this in the future.

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Notation For the convenience of the reader, we give a list of the most common notations used throughout the paper.

$q\colon A \longrightarrow P$	vector bundle or Lie algebroid
$q_*\colon A^* \longrightarrow P$	dual vector bundle or Lie algebroid
$p_P\colon TP \longrightarrow P$	tangent bundle of manifold P
$c_P\colon T^*P \longrightarrow P$	cotangent bundle of manifold P
m, n	elements of P
x, y, z	tangent vectors or vector fields on P
ω, θ	cotangent vectors or 1-forms on P
X, Y, Z	elements or sections of A
ϕ, ψ	elements or sections of A^*
ξ, η	tangent vectors or vector fields on A
Φ, Ψ	cotangent vectors or 1-forms on A
$\mathfrak{X}, \mathfrak{Y}$	tangent vectors or vector fields on A^*
$\mathfrak{F}, \mathfrak{S}$	cotangent vectors or 1-forms on A^*
$\mathfrak{f}, \mathfrak{s}$	elements or sections of the dual of $TA \longrightarrow TP$
$I\colon T(A^*) \longrightarrow T^\bullet(A)$	canonical isomorphism (5.3)
$R\colon T^*(A^*) \longrightarrow T^*(A)$	canonical isomorphism (5.5)

2. Differential calculus on Lie algebroids

As a preliminary, in this section we describe the generalization to arbitrary Lie algebroids of the calculuses of differential forms and multivector fields.

Consider a Lie algebroid A on base P with anchor $a\colon A \longrightarrow TP$. The generalization to A of the standard calculus of differential forms has been treated at length in [17, IV §2] and references given there. Here we give only a quick summary. For $k \geq 0$, let $\wedge^k(A^*)$ denote the kth exterior power bundle on P; we identify $\wedge^k(A^*)$ with $\mathrm{Alt}^k(A, P \times \mathbb{R})$. The *exterior derivative*, or *Lie algebroid coboundary*, $d\colon \Gamma(\wedge^k A^*) \longrightarrow \Gamma(\wedge^{k+1} A^*)$ is defined by

$$
\begin{aligned}
d\phi(X_1, \dots, X_{k+1}) \quad = \quad & \sum_{i=1}^{k+1} (-1)^{i+1} a(X_i)(\phi(X_1, \overset{\hat{}}{\dots}, X_{k+1})) \\
& + \sum_{i\ <j} (-1)^{i+j} \phi([X_i, X_j], X_1, \overset{\hat{}}{\dots}\overset{\hat{}}{\dots}, X_{k+1}),
\end{aligned}
$$

for $\phi \in \Gamma(\wedge^k A^*)$, $X_i \in \Gamma A$, $1 \leq i \leq k+1$. For $X \in \Gamma A$ and $k \geq 0$, the *Lie derivative* $L_X\colon \Gamma(\wedge^k A^*) \longrightarrow \Gamma(\wedge^k A^*)$ is defined by

$$L_X(\phi)(Y_1, \dots, Y_k) = a(X)(\phi(Y_1, \dots, Y_k)) - \sum_{i=1}^{k} \phi(Y_1, \dots, [X, Y_i], \dots, Y_k), \tag{1}$$

for $\phi \in \Gamma(\wedge^k A^*)$, $Y_1, \dots, Y_k \in \Gamma A$. The *contraction*, or *interior multiplication*, $\iota_X\colon \Gamma(\wedge^{k+1} A^*) \longrightarrow \Gamma(\wedge^k A^*)$ is defined by

$$\iota_X(\phi)(Y_1, \dots, Y_k) = \phi(X, Y_1, \dots, Y_k),$$

for $\phi \in \Gamma(\wedge^{k+1} A^*)$, $Y_1, \dots, Y_k \in \Gamma A$.

These operators satisfy the following analogues of the standard calculus of differential forms; see [17, IV §2].

$$d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^k \phi \wedge d\psi \tag{2}$$
$$d^2 = 0 \tag{3}$$
$$L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X \tag{4}$$
$$\iota_{[X,Y]} = L_X \circ \iota_Y - \iota_Y \circ L_X \tag{5}$$
$$L_X = d \circ \iota_X + \iota_X \circ d \tag{6}$$
$$L_{fX}(\phi) = f L_X(\phi) + df \wedge \iota_X(\phi) \tag{7}$$

where $X, Y \in \Gamma A$, $f \in C^\infty(P)$, $\phi \in \Gamma(\wedge^k A^*), \psi \in \Gamma(\wedge^m A^*)$. Note that in (7), $df \in \Gamma(\wedge^1 A^*)$ refers to the Lie algebroid coboundary.

In a similar way the Schouten bracket and Lie derivative of multivector fields extend to A. We refer to a section $D \in \Gamma(\wedge^k A)$ as a *k-multisection* of A. The *generalized Schouten bracket*

$$[\ ,\ ]\colon \Gamma(\wedge^k A) \times \Gamma(\wedge^m A) \longrightarrow \Gamma(\wedge^{k+m-1} A)$$

is characterized by the conditions that $[\ ,\ ]\colon \Gamma(\wedge^1 A) \times \Gamma(\wedge^1 A) \longrightarrow \Gamma(\wedge^1 A)$ coincide with the Lie algebroid bracket, that $[X, f] = a(X)(f)$ for $X \in \Gamma A$, $f \in C^\infty(P)$, and that the properties

$$[D, D'] = -(-1)^{(k-1)(m-1)}[D', D], \tag{8}$$

$$(-1)^{(k-1)(n-1)}[[D, D'], D''] + (-1)^{(m-1)(k-1)}[[D', D''], D] + (-1)^{(n-1)(m-1)}[[D'', D], D'] = 0, \tag{9}$$

$$[D, D' \wedge D''] = [D, D'] \wedge D'' + (-1)^{k(n-1)} D' \wedge [D, D''], \tag{10}$$

hold for all $D \in \Gamma(\wedge^k A)$, $D' \in \Gamma(\wedge^m A)$, $D'' \in \Gamma(\wedge^n A)$. (Compare [12], [13].)

We denote the pairing between a k-form ϕ and a k-multisection D of A variously by $\phi \cdot D$ or $\langle D, \phi \rangle$, according to whichever is clearest in the given context. Given $\phi \in \Gamma(\wedge^k A^*)$, there is a *contraction*, $\iota_\phi\colon \Gamma(\wedge^m A) \longrightarrow \Gamma(\wedge^{m-k} A)$, where $m \geq k$, defined by

$$(\iota_\phi(D))(\psi) = (\phi \wedge \psi) \cdot D.$$

For $X \in \Gamma A$ and $D \in \Gamma(\wedge^k A)$ we also write

$$L_X(D) = [X, D];$$

this is the *Lie derivative* of multisections. It follows from (8)—(10) above that $L_X : \Gamma(\wedge^k A) \longrightarrow \Gamma(\wedge^k A)$ has the following properties:

$$L_X(D \wedge D') = L_X(D) \wedge D' + D \wedge L_X(D') \tag{11}$$
$$[L_X, L_Y] = L_{[X,Y]} \tag{12}$$
$$L_X(fD) = fL_X(D) + a(X)(f)D \tag{13}$$
$$L_{fX}(D) = fL_X(D) - X \wedge \iota_{df}(D) \tag{14}$$

where $X, Y \in \Gamma A,\ D \in \Gamma(\wedge^k A), D' \in \Gamma(\wedge^{k'} A), f \in C^\infty(P)$.

Finally, for $\phi \in \Gamma(\wedge^k A^*), D \in \Gamma(\wedge^k A), X \in \Gamma A$, we have

$$L_X(\phi \cdot D) = L_X(\phi) \cdot D + \phi \cdot L_X(D). \tag{15}$$

This follows from (1) and $L_X(Y_1 \wedge \ldots \wedge Y_k) = \sum_{i=1}^k Y_1 \wedge \ldots \wedge [X, Y_i] \wedge \ldots \wedge Y_k$.

Example 2.1. Let P be a Poisson manifold with Poisson tensor π. It is well-known that the cotangent bundle $T^*P \longrightarrow P$ carries a natural Lie algebroid structure [4], [5]. Given $f \in C^\infty(P)$, denote the Hamiltonian vector field corresponding to f by X_f. Then the anchor $\pi^\# : T^*P \longrightarrow TP$ is determined by $\pi^\#(f\delta g) = fX_g$. Given $\omega, \theta \in \Omega^1(P)$, and writing $X_\omega = \pi^\#\omega$ and $X_\theta = \pi^\#\theta$, the Lie algebroid bracket is

$$\{\omega, \theta\} = L_{X_\omega}\theta - L_{X_\theta}\omega - \delta(\pi(\omega, \theta)).$$

With respect to this structure, the Lie derivatives with respect to $\omega \in \Omega^1(P)$ of a function $f \in C^\infty(P)$, a 1-form θ, and a vector field $X \in \Gamma TP$, are given by

$$L_\omega(f) = X_\omega(f), \qquad L_\omega(\theta) = \{\omega, \theta\}, \qquad (L_\omega X) \cdot \theta = X_\omega(X \cdot \theta) - X \cdot \{\omega, \theta\}.$$

Furthermore, the exterior derivative $d : \Gamma(\wedge^k TP) \longrightarrow \Gamma(\wedge^{k+1} TP)$ is exactly the differential operator of the Poisson complex [14], namely $d(D) = [\pi, D]$, where the bracket is the Schouten bracket in T^*P.

3. Lie bialgebroids

In this section, we introduce the concept of Lie bialgebroid, and develop its elementary properties. The definition which follows uses the bracket on ΓA and the coboundary $d_* : \Gamma(\wedge^k A) \longrightarrow \Gamma(\wedge^{k+1} A)$ associated to A^*; we will see at the end of the section that this is equivalent to the corresponding condition with the roles of A and A^* reversed.

Definition 3.1. Suppose that $A \longrightarrow P$ is a Lie algebroid, and that its dual bundle $A^* \longrightarrow P$ also carries a Lie algebroid structure. Then (A, A^*) is a *Lie bialgebroid* if for any $X,\ Y \in \Gamma(A)$,

$$d_*[X, Y] = L_X d_* Y - L_Y d_* X. \tag{16}$$

Remark 3.2. It is clear that Equation (16) is equivalent to requiring that the coboundary $d_* : \Gamma(A) \longrightarrow \Gamma(\wedge^2 A)$ be a Lie algebra 1-cocycle for the infinite dimensional Lie algebra $\Gamma(A)$, with the module structure on $\Gamma(\wedge^2 A)$ being the natural one. However, since d_* is not $C^\infty(P)$-linear, it is not a Lie algebroid cocycle in the sense of [17].

It is obvious that a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ in the sense of Drinfel'd [8] is a special case of a Lie bialgebroid. Another interesting example is the following.

Example 3.3. Suppose that P is a Poisson manifold. Let A be the usual tangent bundle Lie algebroid $TP \longrightarrow P$, and let $A^* = T^*P \longrightarrow P$ be equipped with the canonical Lie algebroid structure induced from the Poisson struture on P. Clearly (TP, T^*P) is a Lie bialgebroid. In fact, the compatibility condition (16) follows from the graded Jacobi identity of the Schouten brackets [12].

In what follows, we always use a and a_* to denote the anchor maps of A and A^* respectively.

Proposition 3.4. *Assume that (A, A^*) is a Lie bialgebroid. Then, for any $X \in \Gamma(A)$ and $f \in C^\infty(P)$,*

$$L_{df}X = -[d_*f, X].$$

Proof. For any $X, Y \in \Gamma(A)$ and $f \in C^\infty(P)$ we have, by Equation (16), that

$$\begin{aligned} d_*[X, fY] &= d_*(f[X,Y]) + d_*((a(X)f)Y) \\ &= d_*f \wedge [X,Y] + f(-L_Y d_*X + L_X d_*Y) + d_*(a(X)f) \wedge Y \\ &\quad + (a(X)f)d_*Y. \end{aligned}$$

On the other hand,

$$\begin{aligned} d_*[X, fY] &= -L_{fY}d_*X + L_X d_*(fY) \\ &= -L_{fY}d_*X + (L_X d_*f) \wedge Y + (d_*f) \wedge L_X Y + f L_X d_*Y \\ &\quad + (a(X)f)d_*Y. \end{aligned}$$

It thus follows that

$$-L_{fY}d_*X + (L_X d_*f) \wedge Y = f(-L_Y d_*X) + d_*(a(X)f) \wedge Y.$$

According to Equation (14),

$$L_{fY}d_*X = fL_Y d_*X - Y \wedge \iota_{df} d_*X,$$

and so

$$(L_X(d_*f)) \wedge Y = [\iota_{df} d_*X + d_*(a(X)f)] \wedge Y,$$

which implies, using (6), that

$$L_{df}X = -[d_*f, X].$$

□

Corollary 3.5. *Let $\pi_P^\#$ denote the composition $a \circ a_*^* : T^*P \longrightarrow TP$. Then we have*

$$[d_*g, d_*f] = d_*(\pi_P^\#(\delta g)f). \tag{17}$$

Proof. By letting $X = d_*g$ in Proposition 3.4, it follows that

$$[d_*g, d_*f] = L_{df}d_*g = d_*[\iota_{df}(d_*g)] = d_*((a^*\delta f)\cdot(a_*^*\delta g)) = d_*(\pi_P^\#(\delta g)f).$$

□

Proposition 3.6. *Suppose that (A, A^*) is a Lie bialgebroid. Then $\pi_P^\# : T^*P \longrightarrow TP$ defines a Poisson structure on P, and so also does $\bar{\pi}_P^\# = a_* \circ a^* : T^*P \longrightarrow TP$. Moreover, $\pi_P^\#$ and $\bar{\pi}_P^\#$ are opposite to one another.*

Proof. As a first step, we need to prove that $\pi_P^\#$ is skew-symmetric. It follows from Corollary 3.5 that $d_*((\pi_P^\#\delta f)f) = 0$ for any $f \in C^\infty(P)$. In particular, $d_*((\pi_P^\#\delta f^2)f^2) = 0$. It follows immediately that

$$(\pi_P^\#(\delta f)f)d_* f^2 = 0. \tag{18}$$

Applying a to both sides of Equation (18), one obtains that $2f(\pi_P^\#(\delta f)f)^2 = 0$, which yields that $\pi_P^\#(\delta f)f = 0$. This implies immediately that $\pi_P^\#$ is skew-symmetric, since $\pi_P^\#$ is a bundle map. Let $\{g, f\} = (\pi_P^\#\delta g)f$ as usual. Applying a to both sides of Equation (17), we have

$$\pi_P^\#\delta(\{g,f\}) = [\pi_P^\#\delta g, \pi_P^\#\delta f],$$

which is equivalent to the Jacobi identity of the bracket. So $\pi_P^\#$ defines a Poisson structure on P. The rest of the theorem follows immediately. □

To avoid confusion in the sequel, by the Poisson structure induced on P by a Lie bialgebroid (A, A^*), we always mean the structure defined by $\pi_P^\#$, unless explicitly stated otherwise. It follows immediately from Corollary 3.5 that $a_*^*\colon T^*P \longrightarrow A$ is bracket-preserving and a Lie algebroid morphism. In terms of the tangent Poisson structures of [6], we have the following

Corollary 3.7. *Let (A, A^*) be a Lie bialgebroid, and let TP be equipped with the tangent Poisson structure. Then the anchor $a_* : A^* \longrightarrow TP$ is a Poisson map, while $a : A \longrightarrow TP$ is anti-Poisson.*

The next result is dual to, and follows from, Proposition 3.4.

Proposition 3.8. *Assume that (A, A^*) is a Lie bialgebroid. For any $\phi \in \Gamma(A^*)$ and $f \in C^\infty(P)$, we have:*

$$L_{d_* f}\phi = -[df, \phi].$$

Proof. For any $X \in \Gamma(A)$, using (15) twice,

$$\begin{aligned}
(L_{d_* f}\phi)\cdot X &= L_{d_* f}(\phi\cdot X) - \phi\cdot(L_{d_* f}X)\\
&= ((a\circ a_*^*)(\delta f))(\phi\cdot X) - \phi\cdot(L_{d_* f}X)\\
&= -((a_*\circ a^*)(\delta f))(\phi\cdot X) - \phi\cdot(L_{d_* f}X)\\
&= -L_{df}(\phi\cdot X) - \phi\cdot(L_{d_* f}X)\\
&= -[df,\phi]\cdot X - \phi\cdot(L_{df}X + [d_* f, X])\\
&= -[df,\phi]\cdot X.
\end{aligned}$$

In the third equality we used the fact that the bundle map $a\circ a_*^* : T^*P \longrightarrow TP$ is skew-symmetric. □

Corollary 3.9. *Let (A, A^*) be a Lie bialgebroid. Let $X \in \Gamma(A)$, $\phi \in \Gamma(A^*)$, and $\omega \in \Omega^1(P)$. Then*

$$L_{a^*\omega}X = -[a_*^*\omega, X] - a_*^*(\iota_{a(X)}\delta\omega),$$

and

$$L_{a_*^*\omega}\phi = -[a^*\omega, \phi] - a^*(\iota_{a_*(\phi)}\delta\omega).$$

Proof. We shall only prove the first identity. The second one can be shown similarly.

Without loss of generality, we assume that $\omega = g\delta f$ for some $f,\ g \in C^\infty(P)$. Then the left hand side is

$$L_{gdf}X = gL_{df}X + d_*g \wedge (\iota_{df}X) = -g[d_*f, X] + (a(X)f)d_*g,$$

using Proposition 3.4 and (7), while the right hand side is

$$-[a_*^* g\delta f, X] - a_*^*(\iota_{a(X)}(\delta g \wedge \delta f)) = (a(X)f)d_*g - g[d_*f, X].$$

□

We are now ready to prove the following duality theorem for Lie bialgebroids.

Theorem 3.10. *If (A, A^*) is a Lie bialgebroid, then so is (A^*, A).*

To prove this theorem, we need the following lemma:

Lemma 3.11. *For any $X \in \Gamma(A)$, $\phi \in \Gamma(A^*)$ and $f \in C^\infty(P)$,*

$$L_{L_\phi X}f = L_\phi L_X f - X \cdot [\phi, df].$$

Proof. This is a simple calculation using the fact that $L_Z(f) = df \cdot Z$ for all $Z \in \Gamma A$:

$$L_{L_\phi X}f = (L_\phi X)\cdot df = L_\phi(X \cdot df) - X \cdot L_\phi(df) = L_\phi L_X f - X \cdot [\phi, df].$$

□

Proof of Theorem 3.10. Let

$$K = (L_X d_*Y - L_Y d_*X - d_*[X,Y])(\phi \wedge \psi) - (L_\phi d\psi - L_\psi d\phi - d[\phi,\psi])(X \wedge Y).$$

To prove the theorem, it suffices to show that $K = 0$. It follows from a straightforward computation that

$$\begin{aligned}
K &= L_{L_\psi X}(\phi\cdot Y) - L_{L_\psi Y}(\phi\cdot X) - L_\psi[L_X(\phi\cdot Y) - L_Y(\phi\cdot X)]\\
&\quad - L_{L_\phi X}(\psi\cdot Y) + L_{L_\phi Y}(\psi\cdot X) + L_\phi[L_X(\psi\cdot Y) - L_Y(\psi\cdot X)]\\
&\quad - L_{L_Y\phi}(\psi\cdot X) + L_{L_Y\psi}(\phi\cdot X) + L_Y[L_\phi(\psi\cdot X) - L_\psi(\phi\cdot X)]\\
&\quad + L_{L_X\phi}(\psi\cdot Y) - L_{L_X\psi}(\phi\cdot Y) - L_X[L_\phi(\psi\cdot Y) - L_\psi(\phi\cdot Y)]\\
&= -X\cdot[\psi, d(\phi\cdot Y)] + Y\cdot[\psi, d(\phi\cdot X)] + X\cdot[\phi, d(\psi\cdot Y)] - Y\cdot[\phi, d(\psi\cdot X)]\\
&\quad + \phi\cdot[Y, d_*(\psi\cdot X)] - \psi\cdot[Y, d_*(\phi\cdot X)] - \phi\cdot[X, d_*(\psi\cdot Y)]\\
&\quad + \psi\cdot[X, d_*(\phi\cdot Y)]\\
&= L_{d(\phi\cdot Y)}(\psi\cdot X) - L_{d(\phi\cdot X)}(\psi\cdot Y) - L_{d(\psi\cdot Y)}(\phi\cdot X) + L_{d(\psi\cdot X)}(\phi\cdot Y)\\
&= 0.
\end{aligned}$$

Here, in the second step, we have used Lemma 3.11 and its dual, while the third step depends on Propositions 3.4 and 3.8 in the forms $L_{df}(X) = -L_{d_*f}(X)$, $L_{df}(\phi) = -L_{d_*f}(\phi)$. The last step uses $L_{df}(g) = -L_{dg}(f)$, which follows from Proposition 3.6. □

4. Triangular Lie bialgebroids

In this section we briefly consider the special case of Lie bialgebroids defined by a suitable bundle map $A^* \longrightarrow A$. This includes the case of triangular Lie bialgebras and the Lie bialgebroids associated to Poisson manifolds. Throughout this section we let $A \longrightarrow P$ be a Lie algebroid with anchor a, and let $\Lambda \in \Gamma(\wedge^2 A)$ be a bi-section of A satisfying $[\Lambda, \Lambda] = 0$. Let $\Lambda^{\#} : A^* \longrightarrow A$ denote the bundle map associated to Λ, and let $a_* = a \circ \Lambda^{\#} : A^* \longrightarrow TP$. Define a bracket of sections of A^* by

$$[\phi, \psi] = L_{\Lambda^{\#}\phi}\psi - L_{\Lambda^{\#}\psi}\phi - d(\Lambda(\phi, \psi)) = \iota_{\Lambda^{\#}\phi}(d\psi) - \iota_{\Lambda^{\#}\psi}(d\phi) + d(\Lambda(\phi, \psi)) \tag{19}$$

for any $\phi, \psi \in \Gamma(A^*)$.

Theorem 4.1. *The vector bundle $A^* \longrightarrow P$ together with the bracket above and the anchor a_* is a Lie algebroid. Moreover, (A, A^*) is a Lie bialgebroid.*

Proof. For any $\phi, \psi \in \Gamma(A^*)$, and $f \in C^\infty(P)$,

$$\begin{aligned}
[\phi, f\psi] &= L_{\Lambda^{\#}\phi} f\psi - L_{\Lambda^{\#}(f\psi)}\phi - d(\Lambda(\phi, f\psi)) \\
&= f(L_{\Lambda^{\#}\phi}\psi - L_{\Lambda^{\#}\psi}\phi - d(\Lambda(\phi, \psi))) + (L_{\Lambda^{\#}\phi} f)\psi \\
&= f[\phi, \psi] + (a(\Lambda^{\#}\phi)(f))\psi \\
&= f[\phi, \psi] + (a_*(\phi) f)\psi.
\end{aligned}$$

The Jacobi identity of the bracket can be checked directly (compare [4], [11]). Therefore A^* is a Lie algebroid.

The compatiblity between A and A^* follows from the graded Jacobi identity of the Schouten brackets on $\Gamma(\wedge^* A)$ and the following lemma. ☐

Lemma 4.2. *For any $X \in \Gamma(A)$,*

$$d_* X = [\Lambda, X].$$

Proof. For any ϕ, $\psi \in \Gamma(A^*)$, we have

$$\begin{aligned}
(d_* X)(\phi, \psi) &= a_*(\phi)(X \cdot \psi) - a_*(\psi)(X \cdot \phi) - X \cdot [\phi, \psi] \\
&= a(\Lambda^{\#}\phi)(X \cdot \psi) - a(\Lambda^{\#}\psi)(X \cdot \phi) \\
&\quad - X \cdot (L_{\Lambda^{\#}\phi}\psi - L_{\Lambda^{\#}\psi}\phi - d(\Lambda(\phi, \psi))) \\
&= L_{\Lambda^{\#}\phi}(X \cdot \psi) - L_{\Lambda^{\#}\psi}(X \cdot \phi) - X \cdot L_{\Lambda^{\#}\phi}\psi + X \cdot L_{\Lambda^{\#}\psi}\phi \\
&\quad + X \cdot d(\Lambda(\phi, \psi)) \\
&= (L_{\Lambda^{\#}\phi} X) \cdot \psi - (L_{\Lambda^{\#}\psi} X) \cdot \phi + a(X)(\Lambda(\phi, \psi)) \\
&= -L_X(\Lambda^{\#}\phi) \cdot \psi + L_X(\Lambda^{\#}\psi) \cdot \phi + L_X(\Lambda(\phi, \psi) \\
&= \Lambda(\phi, L_X\psi) - \Lambda(\psi, L_X\phi) + L_X(\Lambda(\psi, \phi)) \\
&= -(L_X\Lambda)(\phi, \psi) \\
&= [\Lambda, X](\phi, \psi).
\end{aligned}$$

☐

We shall call such a Lie bialgebroid, constructed as above, a *triangular Lie bialgebroid*. Clearly, triangular Lie bialgebras and the Lie bialgebroids associated to Poisson manifolds as in Example 3.3 are special cases of triangular Lie bialgebroids.

An important property of a triangular Lie bialgebroid is the following (compare a similar result for triangular Lie bialgebras [22], [25]):

Theorem 4.3. *Let (A, A^*) be a triangular Lie bialgebroid. Then $\Lambda^\# : A^* \longrightarrow A$ is a Lie algebroid morphism.*

Proof. Since $\Lambda^\#$ is skew-symmetric, for any $\phi, \psi, \gamma \in \Gamma(A^*)$ we have

$$\begin{aligned}
\langle \Lambda^\#\{\phi, \psi\}, \gamma\rangle &= -\langle L_{\Lambda^\#\phi}\psi, \Lambda^\#\gamma\rangle + \langle L_{\Lambda^\#\psi}\phi, \Lambda^\#\gamma\rangle + \langle d(\Lambda(\phi, \psi)), \Lambda^\#\gamma\rangle \\
&= -L_{\Lambda^\#\phi}(\psi \cdot \Lambda^\#\gamma) + \psi \cdot [\Lambda^\#\phi, \Lambda^\#\gamma] + L_{\Lambda^\#\psi}(\phi \cdot \Lambda^\#\gamma) \\
&\quad - \phi \cdot [\Lambda^\#\psi, \Lambda^\#\gamma] + [a(\Lambda^\#\gamma)](\Lambda(\phi, \psi)) \\
&= [\Lambda, \Lambda](\phi, \psi, \gamma) + \gamma \cdot [\Lambda^\#\phi, \Lambda^\#\psi] \\
&= \langle [\Lambda^\#\phi, \Lambda^\#\psi], \gamma\rangle.
\end{aligned}$$

Therefore $\Lambda^\#$ is a Lie algebroid morphism. □

That (19) defines a Lie algebroid structure on A^* if $[\Lambda, \Lambda] = 0$ was also shown by Kosmann-Schwarzbach and Magri [12, §6.5].

5. Tangent Lie algebroids and their duals

Lie bialgebras first arose as infinitesimal invariants of Poisson Lie groups. In order to study the corresponding relationship between Poisson groupoids and Lie bialgebroids, we need to work extensively with the tangents of vector bundles and Lie algebroids. There are two main reasons for this. Firstly, whereas the (co)tangent bundle of a Lie group is trivializable, this is not usually true of Lie groupoids, and in place of Lie bialgebras of the form $\mathfrak{g} \oplus \mathfrak{g}^*$ we need to consider the cotangent bundles of general Lie algebroids. Secondly, the nonlinearity referred to in the remark following (3.1) can be removed by lifting to the tangent bundle. In this section, we set out the basic facts which we need about tangent vector bundles and their duals.

We refer to §1 of [18], and references given there, for background on double vector bundles, and in particular for the concepts of core and of core sequences. We will be mostly concerned with the tangent double vector bundle of a vector bundle, and various double vector bundles associated to it.

Consider a vector bundle $q\colon A \longrightarrow P$. The *tangent double vector bundle*

$$\begin{array}{ccc}
TA & \xrightarrow{T(q)} & TP \\
{\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle p} \\
A & \xrightarrow[q]{} & P
\end{array} \qquad (20)$$

is described in [3], [20], and [18], and we briefly recall the structure here. In A and TP, we use standard notation. The zero of A over $m \in P$ is 0^A_m, and the zero of TP over m is 0^T_m.

We denote elements of TA by $\xi, \eta, \zeta \ldots$, and we write $(\xi; X, x; m)$ to indicate that $X = p_A(\xi)$, $x = T(q)(\xi)$, and $m = p(T(q)(\xi)) = q(p_A(\xi))$. With respect to the tangent bundle structure (TA, p_A, A), we use standard notation: $+$ for addition, $-$ for subtraction and juxtaposition for scalar multiplication. The notation $T_X(A)$ will always denote the fibre $p_A^{-1}(X)$, for $X \in A$, with respect to this bundle. The zero element in $T_X(A)$ is denoted $\widetilde{0}_X$. We refer to this bundle structure as the p_A*-bundle structure.*

With respect to the $T(q)$*-bundle structure*, $(TA, T(q), TP)$, we use $+\!\!\!+$ for addition, $-$ for subtraction, and $\boldsymbol{.}$ for scalar multiplication. This addition and scalar multiplication on TA are precisely the tangents of the addition and scalar multiplication in A. The fibre over $x \in TP$ will always be denoted $T(q)^{-1}(x)$, and the zero element of this fibre is $T(0)(x)$. If we consider elements ξ of TA as derivatives of paths in A and write

$$\xi = \frac{d}{dt}X_t\Big|_0,$$

where X_t denotes a path in A defined in a neighbourhood of $0 \in \mathbb{R}$, then $p_A(\xi) = X_0$ and $T(q)(\xi) = \frac{d}{dt}q(X_t)\big|_0$. If $\xi, \eta \in TA$ have $T(q)(\xi) = T(q)(\eta)$, then we can arrange that $\xi = \frac{d}{dt}X_t\big|_0$ and $\eta = \frac{d}{dt}Y_t\big|_0$, where $q(X_t) = q(Y_t)$ for all t in a neighbourhood of $0 \in \mathbb{R}$ and then

$$\xi +\!\!\!+ \eta = \frac{d}{dt}(X_t + Y_t)\Big|_0, \quad \lambda \,\boldsymbol{.}\, \xi = \frac{d}{dt}\lambda X_t\Big|_0.$$

For each $m \in P$, the tangent space $T_{0_m}(A_m)$ identifies canonically with A_m; we denote the element of $T_{0_m}(A_m)$ corresponding to $X \in A_m$ by $\overline{X}$. This identifies A with the core of TA. Note that, for $X, Y \in A_m$ and $t \in \mathbb{R}$,

$$\overline{X} + \overline{Y} = \overline{X + Y} = \overline{X} +\!\!\!+ \overline{Y}, \qquad t\overline{X} = \overline{tX} = t \,\boldsymbol{.}\, \overline{X}.$$

Given a morphism of vector bundles $\phi\colon A' \longrightarrow A$, $f\colon P' \longrightarrow P$, we denote the pullback of A across f by $f^!A$, and the induced morphism $A' \longrightarrow f^!A$ over P' by $\phi^!$. Associated to the double vector bundle structure on TA are the two core sequences:

$$q^!A \overset{\tau}{\rightarrowtail} TA \overset{T(q)^!}{\twoheadrightarrow} q^!TP, \tag{21}$$

where the vector bundles have base A, and

$$p^!A \overset{\upsilon}{\rightarrowtail} TA \overset{p_A^!}{\twoheadrightarrow} p^!A, \tag{22}$$

where the vector bundles have base TP. Here τ is the map

$$\tau(X, Y) = \widetilde{0}_X +\!\!\!+ \overline{Y},$$

which assigns to $(X, Y) \in A_m \times A_m$ the element of $T_X(A_m)$ which has its tail at X, and is parallel to Y, and υ is the map

$$\upsilon(x, Y) = T(0)(x) + \overline{Y}.$$

We refer to (21) as the *core sequence for* p_A, and to (22) as the *core sequence for* $T(q)$. We call τ and υ *translation maps.*

Given $X \in \Gamma(A)$ define a vector field $\check{X}$ on A by $\check{X}(Y) = \tau(Y, X(qY))$, $Y \in A$. Then

$$\check{X}(F)(Y) = \frac{d}{dt}F(Y + tX(qY))\Big|_0$$

for $F \in C^\infty(A), Y \in A$, and so

$$\check{X}(f \circ q) = 0, \qquad \check{X}(l_\phi) = \langle \phi, X\rangle \circ q, \qquad [\check{X}, \check{Y}] = 0, \tag{23}$$

for $f \in C^\infty(P)$, $X, Y \in \Gamma(A)$, $\phi \in \Gamma(A^*)$. Here $l_\phi \in C^\infty(A)$ is the function $X \mapsto \langle \phi, X\rangle$. Note also that $(fX)\check{} = (f \circ q)\check{X}$.

A section $X \in \Gamma(A)$ also induces a section $\widehat{X}$ of $T(q)$ by $\widehat{X}(x) = \upsilon(x, X(px))$. Note that

$$(X+Y)\widehat{\ } = \widehat{X} \mathbin{+\!\!\!+} \widehat{Y}, \qquad (fX)\widehat{\ } = (f\circ p) \mathbin{\bullet} \widehat{X},$$

for $X, Y \in \Gamma(A), f \in C^\infty(P)$.

If A is a trivial vector bundle $P\times V \longrightarrow P$, where V is a vector space, we denote elements of $TA = TP\times V\times V$ by $\xi = (x, X, X')$, where if m_t is a path in P with $x = \frac{d}{dt}m_t\big|_0$, we identify (x, X, X') with $\frac{d}{dt}(m_t, X + tX')\big|_0$. The operations in the bundle $T(q)\colon TP\times V\times V \longrightarrow TP$, $(x, X, X') \mapsto x$, are then given by

$$\begin{aligned} (x, X, X') \mathbin{+\!\!\!+} (x, Y, Y') &= (x, X+Y, X'+Y'),\\ t \mathbin{\bullet} (x, X, X') &= (x, tX, tX'),\\ T(0)(x) &= (x, 0, 0), \end{aligned} \tag{24}$$

where $x \in TP$, $X, X', Y, Y' \in V$, $t \in \mathbb{R}$. The operations in the tangent bundle $p_{P\times V}\colon TP\times V\times V \longrightarrow P\times V$, $(x, X, X') \mapsto (px, X)$, are of course given by

$$\begin{aligned} (x, X, X') + (y, X, Y') &= (x+y, X, X'+Y'),\\ t(x, X, X') &= (tx, X, tX'),\\ \widetilde{0}_{(m,X)} &= (0^T_m, X, 0), \end{aligned} \tag{25}$$

where $X, X', Y, Y' \in V$, $t \in \mathbb{R}$ and $x, y \in TP$ have $p(x) = p(y)$. Given $(m, X) \in P\times V$, the corresponding core element of $TP\times V\times V$ is

$$\overline{(m, X)} = (0^T_m, 0, X).$$

Consider now the case of the double tangent bundle T^2P, where $A = TP$. Given a function f on P we define a function $\widetilde{f}$ on TP by $\widetilde{f}(X) = X(f)$, and given a vector field X on P we define a vector field $\widetilde{X}$ on TP by $\widetilde{X} = J\circ T(X)$, where J is the canonical involution of T^2P. The following identities are easily established:

$$\begin{aligned} \widetilde{X+Y} &= \widetilde{X} + \widetilde{Y}, & \widetilde{fX} &= (f\circ p)\widetilde{X} + \widetilde{f}\check{X},\\ \widetilde{X}(f\circ p) &= X(f)\circ p, & \widetilde{X}(\widetilde{f}) &= \widetilde{X(f)},\\ \widetilde{[X, Y]} &= [\widetilde{X}, \widetilde{Y}], & [\widetilde{X}, \check{Y}] &= [X, Y]\check{\ }, \end{aligned} \tag{26}$$

where $X, Y \in \mathcal{X}(P), f \in C^\infty(P)$.

Now suppose that A is a Lie algebroid on P, with anchor $a\colon A \longrightarrow TP$. Define $a_T = J\circ T(a)\colon TA \longrightarrow T^2P$; then a_T is a morphism of vector bundles over TP. Define a bracket on $\Gamma_{TP}TA$ by the conditions

$$[T(X), T(Y)] = T([X, Y]), \qquad [T(X), \widehat{Y}] = [X, Y]\widehat{\ }, \qquad [\widehat{X}, \widehat{Y}] = 0, \tag{27}$$

for $X, Y \in \Gamma(A)$, and extending over $C^\infty(A)$ by the Lie algebroid condition

$$[\xi, F \mathbin{\bullet} \eta] = F \mathbin{\bullet} [\xi, \eta] \mathbin{+\!\!\!+} a_T(\xi)(F) \mathbin{\bullet} \eta. \tag{28}$$

Theorem 5.1. *Let A be a Lie algebroid on P. Then $a_T = J\circ T(a)\colon TA \longrightarrow T^2P$ and the bracket on $\Gamma_{TP}TA$ defined by Equation* (27) *make TA a Lie algebroid on TP. The bundle projection $p_A\colon TA \longrightarrow A$ is a morphism of Lie algebroids over $p\colon TP \longrightarrow P$.*

Proof. Notice first that $a_T(T(X)) = \widetilde{a(X)}$ and $a_T(\widehat{X}) = a(X)^{\vee}$ for $X \in \Gamma(A)$; the latter uses the fact that the map induced on the cores by $T(a)\colon TA \longrightarrow T^2P$ is $a\colon A \longrightarrow TP$ itself. From these and Equations (26) it follows that $a_T([\xi,\eta]) = [a_T(\xi), a_T(\eta)]$ holds for ξ and η of the form $T(X)$ or $\widehat{X}$. Similarly the Jacobi identity is easily verified on sections of these forms.

To prove the nonlinearity condition (28) for $\xi = T(X), F = f\circ p, \eta = \widehat{Y}$, we have

$$\begin{aligned}
[T(X), (f\circ p)\cdot \widehat{Y}] &= [T(X), (fY)^{\wedge}] \\
&= [X, fY]^{\wedge} \\
&= (f\circ p)\cdot [X,Y]^{\wedge} +\!\!\!+\ (a(X)(f)\circ p)\cdot \widehat{Y} \\
&= (f\circ p)\cdot [T(X),\widehat{Y}] +\!\!\!+\ \widetilde{a(X)}(f\circ p)\cdot \widehat{Y} \\
&= (f\circ p)\cdot [T(X),\widehat{Y}] +\!\!\!+\ a_T(T(X))(f\circ p)\cdot \widehat{Y}.
\end{aligned}$$

A similar proof applies for $\xi = \widehat{X}, F = f\circ p, \eta = \widehat{Y}$. The remaining cases are handled by using the equation

$$T(fY) = (f\circ p)\cdot T(Y) +\!\!\!+\ \widetilde{f}\cdot \widehat{Y}$$

and polarizing with respect to

$$\widetilde{fg} = (f\circ p)\widetilde{g} + \widetilde{f}(g\circ p).$$

This proves Equation (28), and the general cases of $a_T([\xi,\eta]) = [a_T(\xi), a_T(\eta)]$ and the Jacobi identity now follow.

To prove that $p_A\colon TA \longrightarrow A$ is a morphism of Lie algebroids, it suffices [9] to prove that if $\xi,\eta \in \Gamma_{TP}TA$ are projectable with $\xi \sim X,\ \eta \sim Y$, then $[\xi,\eta] \sim [X,Y]$. But the projectable sections of TA are precisely the linear combinations of those of the form $T(X)$ and $\widehat{Y}$, and so the result follows from the defining conditions (27). □

In the case of $A = TP$, this construction equips the bundle $T(p)\colon T^2P \longrightarrow TP$ with a Lie algebroid structure with anchor $J\colon T^2P \longrightarrow T^2P$ and bracket

$$[\xi,\eta] = J[J\xi, J\eta].$$

Remark 5.2. Let G be a Lie groupoid on P. We show in Theorem 7.1 that $T(AG)$, with the structure of Theorem 5.1, is isomorphic to $A(TG)$, the Lie algebroid of the tangent groupoid TG on TP, under a map induced by the canonical involution on T^2G.

We now return to consideration of a general vector bundle (A, q, P). It is a remarkable fact that the cotangent bundle of A has a double vector bundle structure:

$$\begin{array}{ccc}
T^*A & \xrightarrow{\ r\ } & A^* \\
{\scriptstyle c_A}\downarrow & & \downarrow{\scriptstyle q_*} \\
A & \xrightarrow[\ q\ ]{} & P;
\end{array} \tag{29}$$

we call (29) the *cotangent dual of* TA. Here (T^*A, c_A, A) is the standard cotangent bundle of A, and the notation $T^*_X(A)$ will always refer to the fibre with respect to c_A. In this bundle we use standard notation, and denote the zero element of $T^*_X(A)$ by $\widetilde{0}^*_X$.

The map $r\colon T^*A \longrightarrow A^*$ is defined as follows. Take $\Phi \in T^*_X(A)$, where $X \in A_m$. Define

$$r(\Phi)(Y) = \Phi(\tau(X,Y))$$

for $Y \in A_m$. Thus $r(\Phi) \in A^*_m$. Given $\Phi \in T^*_X(A),\ \Psi \in T^*_Y(A)$ with $r(\Phi) = r(\Psi) \in A^*_m$, define $\Phi \mathbin{+\!\!\!+} \Psi \in T^*_{X+Y}(A)$ by

$$(\Phi \mathbin{+\!\!\!+} \Psi)(\xi \mathbin{+\!\!\!+} \eta) = \Phi(\xi) + \Psi(\eta),$$

where $\xi \in T_X(A),\ \eta \in T_Y(A)$, and $T(q)(\xi) = T(q)(\eta)$. Similarly, define

$$(\lambda \,.\, \Phi)(\lambda \,.\, \xi) = \lambda\Phi(\xi),$$

for $\lambda \in \mathbb{R}$ and $\xi \in T_X(A)$. The zero element of $r^{-1}(\phi)$, where $\phi \in A^*_m$, is $\widetilde{0}^r_\phi \in T^*_{0_m}(A)$ defined by

$$\widetilde{0}^r_\phi(T(0)(x) + \overline{Y}) = \phi(Y)$$

for $x \in T_m(P), Y \in A_m$.

It is straightforward to verify that (29) is a double vector bundle, and that its core is T^*P. Given $\omega \in T^*_m(P)$, the corresponding core element $\overline{\omega}$ is $\overline{\omega}(T(0)(x) + \overline{Y}) = \omega(x)$, for $x \in T_m(P),\ Y \in A_m$. The core exact sequence over A is

$$q^!T^*P \rightarrowtail T^*A \xrightarrow{r^!} q^!A^*, \tag{30}$$

and this is the dual of the core exact sequence (21) for TA and p_A. The map $q^!T^*P \to T^*A$ sends $(X,\omega) \in A \underset{P}{\times} T^*P$ to $\widetilde{0}^*_X \mathbin{+\!\!\!+} \overline{\omega}$; this is the pullback of ω to A at the point X. The other core exact sequence is

$$q^!_*T^*P \rightarrowtail T^*A \xrightarrow{c^!} q^!_*A, \tag{31}$$

where each bundle here is over A^*.

Similarly there is a double vector bundle

$$\begin{array}{ccc} T^\bullet A & \xrightarrow{\quad T(q)_* \quad} & TP \\ {\scriptstyle r_\bullet}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ A^* & \xrightarrow[\quad q_* \quad]{} & P, \end{array} \tag{32}$$

formed by dualizing the horizontal structure in (20). We refer to this as the $T(q)$-*dual of* TA; the bundle $(T^\bullet(A), T(q)_*, TP)$ is defined to be the dual of the vector bundle $(TA, T(q), TP)$. The fibre of $T(q)_*$ over $x \in TP$ is denoted $T^\bullet_x(A)$. An element $\mathfrak{f} \in T^\bullet_x(A)$ is now a linear map $T(q)^{-1}(x) \longrightarrow \mathbb{R}$, and addition and scalar multiplication in this bundle is the standard addition and multiplication of linear maps. The zero element in $T^\bullet_x(A)$ is denoted $\widetilde{0}^\bullet_x$. We denote the operations in this vector bundle by the usual symbols.

The map $r_\bullet\colon T^\bullet A \longrightarrow A^*$ is defined as follows. Take $\mathfrak{f}\colon T(q)^{-1}(x) \longrightarrow \mathbb{R}$, where $x \in T_m(P)$, and define

$$r_\bullet(\mathfrak{f})(X) = \mathfrak{f}(T(0)(x) + \overline{X}), \qquad X \in A_m.$$

Given $\mathfrak{f} \in T^\bullet_x(A),\ \mathfrak{s} \in T^\bullet_y(A)$ with $r_\bullet(\mathfrak{f}) = r_\bullet(\mathfrak{s}) = \phi \in A^*_m$, define $\mathfrak{f} \mathbin{+\!\!\!+} \mathfrak{s} \in T^\bullet_{x+y}(A)$ by

$$(\mathfrak{f} \mathbin{+\!\!\!+} \mathfrak{s})(\xi + \eta) = \mathfrak{f}(\xi) + \mathfrak{s}(\eta),$$

where $\xi, \eta \in T(A)$ have $T(q)(\xi) = x,\ T(q)(\eta) = y$ and $p_A(\xi) = p_A(\eta)$.

Similarly, scalar multiplication of $\mathfrak{f}$ as above by $\lambda \in \mathbb{R}$ is given by

$$(\lambda \,.\, \mathfrak{f})(\lambda\xi) = \lambda\mathfrak{f}(\xi), \qquad \xi \in T(q)^{-1}(x).$$

The zero element of $r_\bullet^{-1}(\phi)$, where $\phi \in A^*_m$, is $\widetilde{0}^\bullet_\phi\colon T(q)^{-1}(0^T_m) \to \mathbb{R}$, defined by

$$\widetilde{0}^\bullet_\phi(\widetilde{0}_X \mathbin{+\!\!\!+} \overline{Y}) = \phi(Y), \qquad X, Y \in A_m.$$

The core of $T^\bullet A$ can be canonically identified with A^*, with $\phi \in A^*_m$ corresponding to $\overline{\phi} \in T^\bullet A$ where

$$\overline{\phi}(\widetilde{0}_X \mathbin{+\!\!\!+} \overline{Y}) = \phi(X), \qquad X, Y \in A_m.$$

The core sequence for $T^\bullet A$ over TP is now

$$p^! A^* \overset{(p^!_A)^*}{\rightarrowtail} T^\bullet A \overset{r^!_\bullet = v^*}{\twoheadrightarrow} p^! A^*, \tag{33}$$

and this is the dual of the $T(q)$-sequence (22) for TA. The core sequence over A^* is

$$q^!_* A^* \overset{\chi}{\rightarrowtail} T^\bullet A \overset{T(q)^!_*}{\twoheadrightarrow} q^!_* TP, \tag{34}$$

where $\chi(\phi, \psi) = \widetilde{0}^\bullet_\phi \mathbin{+\!\!\!+} \overline{\psi}$ and $(\widetilde{0}^\bullet_\phi \mathbin{+\!\!\!+} \overline{\psi})(\widetilde{0}_X \mathbin{+\!\!\!+} \overline{Y}) = \phi(Y) + \psi(X)$ for $\phi, \psi \in A^*_m,\ X, Y \in A_m$.

If A is the trivial bundle $P \times V$, we can write $T^\bullet A \longrightarrow TP$ as $TP \times V^* \times V^* \longrightarrow TP$ where $\mathfrak{f} = (x, \phi, \phi')$ acts on $\xi = (x, X, X') \in TP \times V \times V$ by

$$\langle (x, \phi, \phi'), (x, X, X') \rangle = \langle \phi, X \rangle + \langle \phi', X' \rangle.$$

The operations in this bundle are accordingly similar to those given in equations (24) for $TP \times V \times V \longrightarrow TP$.

The operations in $r_\bullet\colon TP \times V^* \times V^* \longrightarrow P \times V^*,\ (x, \phi, \phi') \mapsto (px, \phi')$, are given by

$$\begin{aligned} (x, \phi, \phi') \mathbin{+\!\!\!+} (y, \psi, \phi') &= (x + y, \phi + \psi, \phi'), \\ t \,.\, (x, \phi, \phi') &= (tx, t\phi, \phi'), \\ \widetilde{0}^\bullet_{(m,\phi)} &= (0^T_m, 0, \phi), \end{aligned} \tag{35}$$

where $m \in P,\ \phi, \phi', \psi \in V^*$ and $x, y \in TP$ have $p(x) = p(y)$. Given $(m, \phi) \in P \times V^*$, the corresponding core element is

$$\overline{(m, \phi)} = (0^T_m, \phi, 0).$$

Both (29) and (32) are instances of the general dualization process for double vector bundles of Pradines [21].

Proposition 5.3. *For any vector bundle (A, q, P), the double vector bundle (32) is canonically isomorphic to the tangent double vector bundle of A^*, by an isomorphism preserving the side bundles A^* and TP, and the core A^*.*

Proof. Applying the tangent functor to the canonical pairing $\langle\ ,\ \rangle\colon A^* \underset{P}{\times} A \longrightarrow P \times \mathbb{R}$, we obtain a pairing $T(A^*) \underset{TP}{\times} TA \longrightarrow TP \times T\mathbb{R}$, denoted $\langle\!\langle\ ,\ \rangle\!\rangle$, given explicitly by

$$\langle\!\langle \mathfrak{X}, \xi \rangle\!\rangle = \frac{d}{dt}\langle \phi_t, X_t\rangle\Big|_0, \tag{36}$$

where $\mathfrak{X} = \frac{d}{dt}\phi_t\big|_0 \in T(A^*)$, $\xi = \frac{d}{dt}X_t\big|_0 \in TA$ have $T(q_*)(\mathfrak{X}) = T(q)(\xi)$; we can thus arrange that $q_*(\phi_t) = q(X_t)$ for t near zero. It is straightforward to prove that this is nondegenerate, and therefore gives an isomorphism $I\colon T(A^*) \longrightarrow T^\bullet(A)$ of the vector bundle structures over TP.

We need to prove that I is also a morphism of the vector bundle structures over A^*; that is, that $r_\bullet \circ I = p_{A^*}$, $I(\mathfrak{X} + \mathfrak{Y}) = I(\mathfrak{X}) \mathbin{+\!\!\!+} I(\mathfrak{Y})$ and $I(t\mathfrak{X}) = t \mathbin{.} I(\mathfrak{X})$ for suitable $\mathfrak{X}, \mathfrak{Y} \in T(A^*)$ and $t \in \mathbb{R}$. This is a local question, and so we can assume that A is a trivial bundle $P \times V$. The tangent double vector bundle structure for $A = P \times V$ is given in (24) and (25), and the structure of $T(A^*)$ is similar. Take $\xi = (x, X, X') \in TP \times V \times V$ and $\mathfrak{X} = (x, f, f') \in TP \times V^* \times V^*$. Then

$$\begin{aligned}
\langle\!\langle \mathfrak{X}, \xi \rangle\!\rangle &= \left\langle \frac{d}{dt}(m_t, f + tf')\Big|_0, \frac{d}{dt}(m_t, X + tX')\Big|_0 \right\rangle \\
&= \frac{d}{dt}(f(X) + tf'(X) + f(tX') + t^2 f'(X'))\Big|_0 \\
&= f'(X) + f(X').
\end{aligned}$$

It follows that I is given locally by $I(x, f, f') = (x, f', f)$. Now it is straightforward, comparing equations (25) and (35), to see that I is a vector bundle morphism over A^*.

It is also clear from the local description that I is an isomorphism of double vector bundles, and preserves the side bundles and the core. □

Although $T^\bullet(A)$ thus does not provide a new structure, we will find the alternative description of $T(A^*)$ which it provides useful.

Remark 5.4. Consider the canonical involution $J\colon T^2P \longrightarrow T^2P$. Dualizing over TP we obtain a vector bundle morphism

$$\begin{array}{ccc}
T^\bullet(TP) & \xrightarrow{\ J^*\ } & T^*(TP) \\
{\scriptstyle T(p)_*}\big\downarrow & & \big\downarrow{\scriptstyle c_{TP}} \\
TP & =\!= & TP,
\end{array}$$

which is easily seen to be a morphism of double vector bundles preserving TP and T^*P. Composing with I gives an isomorphism of double vector bundles $T(T^*P) \longrightarrow T^*(TP)$, also preserving the side bundles, which we denote J'. For a local description see, for example, [6].

We now show that $T^*(A^*)$ and $T^*(A)$ are isomorphic as double vector bundles.

Theorem 5.5. *For any vector bundle (A, q, P), the cotangent dual $T^*(A)$ (29) is canonically isomorphic to the cotangent dual $T^*(A^*)$ of A^*, by a double vector bundle isomorphism which preserves the side bundles A and A^*.*

Proof. Write $K = A^* \underset{P}{\times} A$ and define $F\colon K \longrightarrow \mathbb{R}$ to be the pairing. By a construction of Tulczyjew [23], there is a Lagrangian submanifold $N \subseteq T^*(A^* \times A) = T^*(A^*) \times T^*(A)$ defined to consist of all elements $(\mathfrak{F}, \Phi)$, where $(\mathfrak{F}; \psi, Y; m) \in T^*(A^*)$ and $(\Phi; X, \phi; m) \in T^*(A)$ have $(\psi, X) \in A^* \underset{P}{\times} A$, and such that

$$\langle (\mathfrak{F}, \Phi), (\mathfrak{X}, \xi) \rangle = \langle dF, (\mathfrak{X}, \xi) \rangle \tag{37}$$

for all $(\mathfrak{X}, \xi) \in T(A^* \underset{P}{\times} A) = T(A^*) \underset{TP}{\times} T(A)$ compatible with $(\mathfrak{F}, \Phi)$. Here the compatibility condition forces $\mathfrak{X}$ and ξ to have the forms $(\mathfrak{X}; \psi, x : m)$ and $(\xi; X, x; m)$ respectively.

Now the pairing on the left of Equation (37) is simply $\langle \mathfrak{F}, \mathfrak{X} \rangle + \langle \Phi, \xi \rangle$, and $\langle dF, (\mathfrak{X}, \xi) \rangle = \langle\!\langle \mathfrak{X}, \xi \rangle\!\rangle$. Thus the condition (37) becomes

$$\langle \mathfrak{F}, \mathfrak{X} \rangle + \langle \Phi, \xi \rangle = \langle\!\langle \mathfrak{X}, \xi \rangle\!\rangle. \tag{38}$$

We claim that N is the graph of the isomorphism we seek. To prove that it is a graph, we must show that, given $\mathfrak{F} \in T^*(A^*)$, there exists a unique $\Phi \in T^*(A)$ satisfying (38). Since $\mathfrak{F}$ and Φ must lie above the same point $m \in P$, we can work locally. Assume, therefore, that $A = P \times V$ is a trivial bundle, and write

$$\mathfrak{F} = (\chi, \psi, Y) \in T^*P \times V^* \times V; \qquad \Phi = (\omega, X, \phi) \in T^*P \times V \times V^*;$$
$$\mathfrak{X} = (x, \psi, \theta) \in TP \times V^* \times V^*; \qquad \xi = (x, X, Z) \in TP \times V \times V.$$

Then $\langle \mathfrak{F}, \mathfrak{X} \rangle = \langle \chi, x \rangle + \langle \theta, Y \rangle$; $\langle \Phi, \xi \rangle = \langle \omega, x \rangle + \langle \phi, Z \rangle$; $\langle\!\langle \mathfrak{X}, \xi \rangle\!\rangle = \langle \psi, Z \rangle + \langle \theta, X \rangle$, and (38) becomes

$$\langle \chi, x \rangle + \langle \theta, Y \rangle + \langle \omega, x \rangle + \langle \phi, Z \rangle = \langle \psi, Z \rangle + \langle \theta, X \rangle.$$

It is clear that, given χ, ψ and Y, the unique ω, X and ϕ for which this equation is satisfied for all x, θ and Z are $\omega = -\chi$, $X = Y$ and $\phi = \psi$. This proves that N is the graph of a map $R\colon T^*(A^*) \longrightarrow T^*(A)$. It now follows from the local representation $R\colon (\chi, \psi, Y) \mapsto (-\chi, Y, \psi)$ that R is a diffeomorphism of double vector bundles, and preserves A and A^*. ☐

It is clear from the local representation that the map of the cores $T^*P \longrightarrow T^*P$ induced by R is the negative of the identity. Since N is Lagrangian, it also follows that R is anti-symplectic with respect to the canonical symplectic structures on the cotangent bundles.

Lastly in this section, we make precise the relationship between the tangent Lie algebroid of Theorem 5.1 and the tangent Poisson structures of Courant [6]. We first need to recall the duality between Lie algebroids and (what we call) Poisson vector bundles; see [4], [5] for further details.

Let $q\colon A \longrightarrow P$ be a Lie algebroid with anchor $a\colon A \longrightarrow TP$. For any $X \in \Gamma(A)$, write l_X for the corresponding linear function $\phi \mapsto \langle \phi, X \rangle$ on A^*. Then the dual bundle $q_*\colon A^* \longrightarrow P$ has a Poisson structure characterized by the following three equations:

$$\{l_X, l_Y\} = l_{[X,Y]}, \qquad \{l_X, q_*^* f\} = q_*^*(a(X)f), \qquad \{q_*^* f, q_*^* g\} = 0, \tag{39}$$

where $X, Y \in \Gamma(A)$ and $f, g \in C^\infty(P)$. Call this the Poisson structure *dual* to the Lie algebroid structure on A.

Conversely, consider a vector bundle $E \longrightarrow P$. Denote the $C^\infty(P)$-module of functions $E \longrightarrow \mathbb{R}$ which are fibrewise linear by $C^\infty_{\ell in}(E)$. Thus $C^\infty_{\ell in}(E) \cong \Gamma(E^*)$. If E has a Poisson structure such that the Poisson bracket of any two elements of $C^\infty_{\ell in}(E)$ lies in $C^\infty_{\ell in}(E)$, then we call E a *Poisson vector bundle*. Given a Poisson vector bundle $q\colon E \longrightarrow P$, define a bracket on $\Gamma(E^*)$ by $l_{[X,Y]} = \{l_X, l_Y\}$, where $X, Y \in \Gamma(E^*)$. Further, given $X \in \Gamma(E^*)$, the Hamiltonian vector field $H_{l_X} = \{l_X, -\} \in \Gamma TE$ projects under q to a vector field on P which we denote by $a(X)$. The resulting map $a\colon E^* \longrightarrow TP$ and the bracket $[\ ,\ ]$ make E^* a Lie algebroid on P.

Next, recall the tangent Poisson structures of [6]. Given a manifold P, the module of functions on TP is generated by all $\widetilde{f}$ and all $f \circ p$ for $f \in C^\infty(P)$; here $\widetilde{f} = l_{\delta f}$ and $p\colon TP \longrightarrow P$ is the bundle projection. If P has a Poisson bracket $\{\ ,\ \}$ then the tangent Poisson structure on TP is characterized by

$$\{\widetilde{f}, \widetilde{g}\} = \widetilde{\{f, g\}}, \qquad \{\widetilde{f}, g \circ p\} = \{f, g\} \circ p, \qquad \{f \circ p, g \circ p\} = 0, \tag{40}$$

for $f, g \in C^\infty(P)$. This makes $TP \longrightarrow P$ a Poisson vector bundle, and as such it is dual to the Lie algebroid structure on $T^*P \longrightarrow P$ of [4].

Theorem 5.6. (i) *If $E \longrightarrow P$ is a Poisson vector bundle, then the tangent Poisson structure on TE makes $TE \longrightarrow TP$ a Poisson vector bundle.*

(ii) *Let (A, q, P) be a Lie algebroid, and give $TA \longrightarrow TP$ the tangent Lie algebroid structure of Theorem 5.1, and $A^* \longrightarrow P$ the dual Poisson vector bundle structure of* (39)*. Then $I\colon T(A^*) \longrightarrow T^\bullet(A)$ is a Poisson isomorphism with respect to the tangent Poisson structure on $T(A^*)$ induced from the Poisson structure on A^*, and the dual Poisson structure on $T^\bullet(A)$ induced from the tangent Lie algebroid structure on $TA \longrightarrow TP$.*

Proof. The two statements have essentially the same proof, and we formulate it for the second statement. First observe that if $X \in \Gamma(A)$, then $\widetilde{l_X}$ and $l_X \circ p_*$, where $p_*\colon T(A^*) \longrightarrow A^*$ is the bundle projection, are fibrewise linear functions $T(A^*) \longrightarrow \mathbb{R}$ with respect to the bundle $T(A^*) \longrightarrow TP$; further, the $\widetilde{l_X}$ and $l_X \circ p_*$ generate $C^\infty_{\ell in}(T(A^*))$. Now it is easy to verify that, for all $X \in \Gamma(A)$,

$$\widetilde{l_X} = l_{T(X)} \circ I, \qquad l_X \circ p_* = l_{\widehat{X}} \circ I,$$

where l on the right-hand sides refers to the duality between $TA \longrightarrow TP$ and $T^\bullet A \longrightarrow TP$.

Given $X, Y \in \Gamma(A)$, and using (39), (40) and (27),

$$\begin{aligned}\{l_{T(X)} \circ I, l_{T(Y)} \circ I\} &= \{\widetilde{l_X}, \widetilde{l_Y}\} = \widetilde{\{l_X, l_Y\}} = \widetilde{l_{[X,Y]}} \\ &= l_{T([X,Y])} \circ I = l_{[T(X),T(Y)]} \circ I = \{l_{T(X)}, l_{T(Y)}\} \circ I,\end{aligned}$$

and similarly for the other two conditions. □

It is interesting to note that, for a Lie algebroid (A, q, P), the double vector bundle (29) has Poisson structures on both vertical bundles, and Lie algebroid structures on both horizontal bundles, whilst (32) has Poisson structures horizontally and Lie algebroid structures vertically. The relationship between these structures will be dealt with fully elsewhere. Here we just note the following, whose proof is straightforward.

Proposition 5.7. *Let (A, q, P) be a Lie algebroid. Then in the double vector bundle*

$$\begin{array}{ccc} T^*(A^*) & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ A^* & \longrightarrow & P \end{array} \qquad (41)$$

the structure maps for the horizontal vector bundles are Lie algebroid morphisms with respect to the given structure on A and the cotangent structure on $T^*(A^*)$ induced by the Poisson structure on A^*. In particular, the anchor $\pi^{\#}_{A^*}: T^*(A^*) \longrightarrow T(A^*)$ is a double vector bundle morphism with respect to $a\colon A \longrightarrow TP$ and $\mathrm{id}\colon A^* \longrightarrow A^*$, and induces $-a^*\colon T^*P \longrightarrow A^*$ on the cores.

We will see that this structure plays a crucial role when (A, A^*) is a Lie bialgebroid.

6. An equivalent definition

The purpose of this section is to give an equivalent definition, 6.2 below, of Lie bialgebroids in terms of Lie algebroid morphisms. The importance of this will become apparent in §8. In fact, it is this equivalent definition that mainly motivates our present work.

The notion of Lie algebroid morphism, and its properties as an infinitesimal analogue of Lie groupoid morphism, were studied at length in [9]. In applications, the following equivalent definition is useful, particularly when dealing with Poisson geometry.

Proposition 6.1. *Let A and B be Lie algebroids on bases P and Q and consider a vector bundle morphism $F\colon A \longrightarrow B$, $f : P \longrightarrow Q$. Let $\mathcal{C}$ be the submanifold of $A^* \times \overline{B}^*$ consisting of all elements $(\phi, \psi) \in A^* \times \overline{B}^*$ such that $\langle \phi, X\rangle = \langle \psi, F(X)\rangle$ for all $X \in A$ compatible with ϕ. Then F is a Lie algebroid morphism if and only if $\mathcal{C}$ is a coisotropic submanifold of $A^* \times \overline{B}^*$, where A^* and B^* are equipped with the Poisson structures dual to the Lie algebroids A and B, respectively.*

Proof. Evidently $\mathcal{C}$ is the annihilator in $A \times B$ of the graph of (F, f). Now the result follows from the facts that (i), the vector bundle morphism (F, f) is a Lie algebroid morphism if and only if its graph is a Lie subalgebroid of $A \times B$ [9], and (ii), if $C \longrightarrow P$ is a Lie algebroid, and $D \longrightarrow Q$ is a vector subbundle, then D is a Lie subalgebroid of C if and only if the annihilator $D^{\perp}$ of D in C^* is coisotropic in C^*. □

The main theorem of this section is the following.

Theorem 6.2. *Suppose that $q\colon A \longrightarrow P$ is a Lie algebroid such that its dual vector bundle $q_*\colon A^* \longrightarrow P$ also has a Lie algebroid structure. Let a, a_* be their anchors.*

Then (A, A^*) is a Lie bialgebroid if and only if

$$\begin{array}{ccc} T^*(A^*) & \xrightarrow{\quad\Pi\quad} & TA \\ \big\downarrow & & \big\downarrow \\ A^* & \xrightarrow[\quad a_*\quad]{} & TP \end{array} \tag{42}$$

is a Lie algebroid morphism, where the domain $T^*(A^*) \longrightarrow A^*$ is the cotangent Lie algebroid induced by the Poisson structure on A^*, the target $TA \longrightarrow TP$ is the tangent Lie algebroid of A, and $\Pi\colon T^*(A^*) \longrightarrow TA$ is the composition of the isomorphism $R\colon T^*A^* \longrightarrow T^*A$ of Theorem 5.5 with $\pi_A^\# : T^*A \longrightarrow TA$.

The proof of Theorem 6.2 will take us to the end of the section. We assume until then that A and A^* satisfy the hypotheses of 6.2. The three preliminary results which follow are needed in the course of the proof.

Lemma 6.3. *Given $(\xi; X_m, x; m) \in TA$ and $(\mathfrak{X}; \phi_m, x; m) \in T(A^*)$, let $X \in \Gamma(A)$ and $\phi \in \Gamma(A^*)$ be any sections taking the values X_m and ϕ_m at m. Then*

$$\langle\!\langle \mathfrak{X}, \xi \rangle\!\rangle = \mathfrak{X}(l_X) + \xi(l_\phi) - x(\langle \phi, X\rangle).$$

The proof is straightforward. Given $\omega \in T^*_m(P)$ and $X \in A_m$, denote the pullback of ω to A at the point X by $q^*(X, \omega)$.

Proposition 6.4. *For $X \in \Gamma(A)$ and $\phi \in \Gamma(A^*)$,*

$$R(\delta l_X(\phi_m)) = \delta l_\phi(X_m) - q^*(X_m, \delta\langle \phi, X\rangle),$$

where $\delta l_X(\phi_m) \in T^*_{\phi_m}(A^*)$ is the value of the 1-form δl_X on A^* at ϕ_m, and similarly for $\delta l_\phi(X_m)$.

Proof. For any $m \in P$, obviously $(\phi_m, X_m) \in A^* \underset{P}{\times} A = K$, in the notation of the proof of Theorem 5.5. It suffices to show that

$$(\delta l_X(\phi_m),\ \delta l_\phi(X_m) - q^*(X_m, \delta\langle \phi, X\rangle)) \in N.$$

Let $(\mathfrak{X}, \xi)$ be any tangent vector of K at the point (ϕ_m, X_m). Then

$$\langle \mathfrak{X}, \delta l_X(\phi_m)\rangle + \langle \xi, \delta l_\phi(X_m) - q^*(X_m, \delta\langle\phi, X\rangle)\rangle = \\ \mathfrak{X}(l_X)(\phi_m) + \xi(l_\phi)(X_m) - (T(q)(\xi))(\langle \phi, X\rangle),$$

and this is equal to $\langle\!\langle \mathfrak{X}, \xi\rangle\!\rangle$ by Lemma 6.3. The conclusion now follows from the proof of Theorem 5.5. □

Corollary 6.5. *Let $X \in \Gamma(A)$ be any section. Then for any $(\phi_m, \psi_m) \in A^* \underset{P}{\times} A^*$, with $q_*(\phi_m) = q^*(\psi_m) = m$,*

$$\pi_A(R(\delta l_X(\phi_m)),\ R(\delta l_X(\psi_m))) = -(d_*X)(\phi_m \wedge \psi_m),$$

where π_A denotes the Poisson tensor on A induced from the Lie algebroid A^*.

Proof. Let $\phi \in \Gamma(A^*)$ be any section through the point ϕ_m and $\psi \in \Gamma(A^*)$ any section through the point ψ_m. Then,

$$\begin{aligned}
&\pi_A(R(\delta l_X(\phi_m)),\ R(\delta l_X(\psi_m))) \\
&\quad = \pi_A(\delta l_\phi(X_m) - q^*(X_m, \delta\langle\phi, X\rangle), \delta l_\psi(X_m) - q^*(X_m, \delta\langle\psi, X\rangle)) \\
&\quad = \langle[\phi,\psi], X\rangle - L_\phi(\langle\psi, X\rangle) + L_\psi(\langle\phi, X\rangle) \\
&\quad = -(d_*X)(\phi_m \wedge \psi_m),
\end{aligned}$$

which implies the desired result. □

Throughout this section, let $\mathcal{C}$ denote the annihilator of the graph of (Π, a_*). Thus $\mathcal{C}$ is the subset of $TA^* \times TA^*$ consisting of all elements $(\mathfrak{X}, \mathfrak{Y}) \in T_\phi A^* \times T_\psi A^*$ such that $a_*(\phi) = T(q_*)(\mathfrak{Y})$ and $\langle \mathfrak{X}, \mathfrak{F}\rangle = \langle\langle \mathfrak{Y}, \Pi(\mathfrak{F})\rangle\rangle$ for all $\mathfrak{F} \in T^*_\phi A^*$. We first give another description of $\mathcal{C}$, which is more useful in practice. For this purpose, we need to introduce the following function F_X.

For any $X \in \Gamma(A)$, let F_X be the function on $TA^* \times TA^*$ defined by

$$F_X(\mathfrak{X}, \mathfrak{Y}) = \mathfrak{X}(l_X) - \mathfrak{Y}(l_X) + f_{d_*X}(\phi, \psi),$$

where f_{d_*X} is any function on $A^* \times A^*$ extending the function $(\phi, \psi) \mapsto (d_*X)(\phi \wedge \psi)$ on $A^* \underset{P}{\times} A^*$.

Proposition 6.6. *Let* $(\mathfrak{X}, \mathfrak{Y}) \in T_{\phi_m} A^* \times T_{\psi_m} A^*$. *Then* $(\mathfrak{X}, \mathfrak{Y}) \in \mathcal{C}$ *if and only if the three conditions*

$$a_*(\phi_m) = T(q_*)(\mathfrak{Y}), \qquad a_*(\psi_m) = T(q_*)(\mathfrak{X}), \qquad F_X(\mathfrak{X}, \mathfrak{Y}) = 0, \quad X \in \Gamma(A),$$

hold.

We first need a lemma.

Lemma 6.7. *Suppose that* $(\phi_m, \psi_m) \in A^* \underset{P}{\times} A^*$ *with* $m = q_*(\phi_m) = q_*(\psi_m)$. *Take* $\omega \in T^*_m(P)$. *Then for the pullbacks* $q_*^*(\phi_m, \omega)$ *and* $q_*^*(\psi_m, \omega)$ *of* ω *to* $\phi_m, \psi_m \in A^*$, *we have*

$$\pi_A(R(q_*^*(\phi_m, \omega)), R(q_*^*(\psi_m, \omega))) = \langle a_*(\psi_m) - a_*(\phi_m), \omega\rangle.$$

Proof. Recall from the exact sequence (30), applied to A^*, that $q_*^*(\phi_m, \omega) = \widetilde{0}^*_{\phi_m} + \overline{\omega} \in T^*_{\phi_m}(A^*)$. Since R is a double vector bundle morphism reversing the core, it follows that $R(\widetilde{0}^*_{\phi_m} + \overline{\omega}) = \widetilde{0}^r_{\phi_m} - \overline{\omega}$. Now $\widetilde{0}^r_{\phi_m} = \delta l_\phi(0_m)$, where $\phi \in \Gamma(A^*)$ is any section passing through ϕ_m. So

$$\begin{aligned}
\pi_A(R(q_*^*(\phi_m, \omega)), R(q_*^*(\psi_m, \omega))) &= \pi_A(\delta l_\phi(0_m) - \overline{\omega}, \delta l_\psi(0_m) - \overline{\omega}) \\
&= l_{[\phi,\psi]}(0_m) - \langle a_*(\phi), \omega\rangle + \langle a_*(\psi), \omega\rangle,
\end{aligned}$$

whence the result. Note that $\overline{\omega}$ is the pullback of ω to $0^*_m \in A^*_m$. □

Proof of Proposition 6.6. It is clear that $T^*_{\phi_m} A^*$ is spanned by the covectors of the form $(\delta l_X)(\phi_m)$, for $X \in \Gamma(A)$, and those of the form $q_*^*(\phi_m, \omega)$, for $\omega \in T^*_m P$. So $(\mathfrak{X}, \mathfrak{Y}) \in \mathcal{C}$ if and only if the three conditions

$$\begin{aligned}
a_*(\phi_m) &= T(q_*)(\mathfrak{Y}), \\
\langle \mathfrak{X}, (\delta l_X)(\phi_m)\rangle &= \langle\langle \mathfrak{Y}, \Pi(\delta l_X(\phi_m))\rangle\rangle, \\
\langle \mathfrak{X}, q_*^*(\phi_m, \omega)\rangle &= \langle\langle \mathfrak{Y}, \Pi(q_*^*(\phi_m, \omega))\rangle\rangle,
\end{aligned}$$

hold for all $X \in \Gamma(A),\ \omega \in T^*_m(P)$.

Using Lemma 6.3 and letting $\psi \in \Gamma(A^*)$ be any section passing through ψ_m, we have

$$\langle\!\langle \mathfrak{Y}, \Pi(\delta l_X(\phi_m))\rangle\!\rangle = \mathfrak{Y}(l_X) + \Pi(\delta l_X(\phi_m))(l_\psi) - T(q_*)(\mathfrak{Y})(\langle\psi, X\rangle).$$

Now, using Proposition 6.4 and Corollary 6.5,

$$\begin{aligned}\Pi(\delta l_X(\phi_m))(l_\psi) &= \pi_A(R(\delta l_X(\phi_m)), R(\delta l_X(\psi_m))) \\ &\quad + \pi_A(R(\delta l_X(\phi_m)), q^*(X_m, \delta\langle\psi, X\rangle)) \\ &= -d_*X(\phi_m \wedge \psi_m) + T(q_*)(\mathfrak{Y})\langle\psi, X\rangle,\end{aligned}$$

since $a_*(\phi_m) = T(q_*)(\mathfrak{Y})$. Thus the second condition is equivalent to $F_X(\mathfrak{X}, \mathfrak{Y}) = 0$ for all $X \in \Gamma(A)$.

Similarly, using Lemma 6.7,

$$\begin{aligned}\langle\!\langle \mathfrak{Y}, \Pi(q_*^*(\phi_m, \omega))\rangle\!\rangle &= \mathfrak{Y}(l_0) + \Pi((q_*^*(\phi_m, \omega))(l_\psi) - T(q_*)(\mathfrak{Y})(\langle\psi, 0\rangle) \\ &= \pi_A(R(q_*^*(\phi_m, \omega)), \delta l_\psi(0_m) + \overline{\omega}) - \pi_A(R(q_*^*(\phi_m, \omega)), \overline{\omega}) \\ &= \pi_A(R(q_*^*(\phi_m, \omega)), R(q_*^*(\psi_m, \omega)) \\ &\quad - \pi_A(R(q_*^*(\phi_m, \omega)), R(q_*^*(0_m, \omega))) \\ &= \langle a_*(\psi_m) - a_*(\phi_m), \omega\rangle - \langle a_*(0_m) - a_*(\phi_m), \omega\rangle \\ &= \langle a_*(\psi_m), \omega\rangle,\end{aligned}$$

and so the third condition is equivalent to $a_*(\psi_m) = T(q_*)(\mathfrak{X})$. □

The next result is a corollary of Proposition 6.6.

Corollary 6.8. *Let π be the projection of $T^*A^* \times T^*A^*$ onto $A^* \times A^*$. Then $\pi(\mathcal{C}) = A^* \underset{P}{\times} A^*$.*

For any $\omega \in \Omega^1(P)$, define functions G_ω and H_ω on $TA^* \times T\overline{A^*}$ by:

$$G_\omega(\mathfrak{X}, \mathfrak{Y}) = \langle a_*(\phi) - T(q_*)\mathfrak{Y}, \omega\rangle, \qquad \text{and} \qquad H_\omega(\mathfrak{X}, \mathfrak{Y}) = \langle a_*(\psi) - T(q_*)\mathfrak{X}, \omega\rangle,$$

where $\mathfrak{X} \in T^*_\phi(A^*)$, $\mathfrak{Y} \in T^*_\psi(A^*)$. Proposition 6.6 shows that $\mathcal{C}$ is the set of common zeros of the three families of functions $F_X,\ G_\omega, H_\omega$, for $X \in \Gamma(A),\ \omega \in \Omega^1(P)$. The next step is to calculate the Poisson brackets of these functions on $TA^* \times T\overline{A^*}$. Throughout the rest of the section, A^* is understood to have the Poisson structure dual to the Lie algebroid structure on A, the opposite Poisson structure is denoted $\overline{A^*}$, and TA^* and $T\overline{A^*}$ have the corresponding tangent Poisson structures.

Theorem 6.9. *Take $X, Y \in \Gamma(A)$. Then for any $(\mathfrak{X}, \mathfrak{Y}) \in T_\phi A^* \times T_\psi \overline{A^*}$,*

$$\{F_X, F_Y\}(\mathfrak{X}, \mathfrak{Y}) - F_{[X,Y]}(\mathfrak{X}, \mathfrak{Y}) = (L_X d_* Y - L_Y d_* X - d_*[X, Y])(\phi \wedge \psi),$$

where the Poisson bracket on the left hand side is with respect to the product Poisson structure on $TA^* \times T\overline{A^*}$.

Again we start with a lemma. Let $D \in \Gamma(\wedge^2 A)$ be any bi-section of A. Thus D defines a function on $A^* \underset{P}{\times} \overline{A^*}$ by $(\phi, \psi) \mapsto D(\phi \wedge \psi)$. Let f_D be any extension of D to $A^* \times \overline{A^*}$.

Lemma 6.10. *For any $X \in \Gamma(A)$, let l^1_X and l^2_X denote the linear functions on $A^* \times \overline{A^*}$ defined by $l^1_X(\phi, \psi) = \langle\phi, X\rangle$, and $l^2_X(\phi, \psi) = \langle\psi, X\rangle$, for any $(\phi, \psi) \in A^* \times \overline{A^*}$ respectively. Then*

$$\{l^1_X, f_D\}(\phi, \psi) - \{l^2_X, f_D\}(\phi, \psi) = (L_X D)(\phi \wedge \psi), \tag{43}$$

where the Poisson bracket on the left hand side is with respect to the product Poisson structure on $A^* \times \overline{A^*}$.

Proof. Without loss of generality, we assume that $D = Y_1 \wedge Y_2$ for $Y_1,\ Y_2 \in \Gamma(A)$ and $f_D(\phi, \psi) = Y_1(\phi)Y_2(\psi) - Y_1(\psi)Y_2(\phi)$ for $\phi, \psi \in A^*$. Let ϕ and ψ also denote any sections of A^* through the points ϕ and ψ respectively. Then,

$$\begin{aligned}
(L_X D)(\phi \wedge \psi) &= -D(L_X(\phi \wedge \psi)) + L_X(D(\phi \wedge \psi)) \\
&= -(\iota_\phi D) \cdot (L_X \psi) + (\iota_\psi D) \cdot (L_X \phi) + L_X(D(\phi \wedge \psi)) \\
&= [X, \iota_\phi(D)](\psi) - [X, \iota_\psi(D)](\phi) - L_X(D(\phi \wedge \psi)) \\
&= \langle [X, (Y_1 \cdot \phi)Y_2 - (Y_2 \cdot \phi)Y_1], \psi \rangle \\
&\quad - \langle [X, (Y_1 \cdot \psi)Y_2 - (Y_2 \cdot \psi)Y_1], \phi \rangle \\
&\quad - L_X(D(\phi \wedge \psi)) \\
&= (Y_1 \cdot \phi)([X, Y_2] \cdot \psi) - (Y_2 \cdot \phi)([X, Y_1] \cdot \psi) \\
&\quad - (Y_1 \cdot \psi)([X, Y_2] \cdot \phi) + (Y_2 \cdot \psi)([X, Y_1] \cdot \phi).
\end{aligned}$$

On the other hand, it can be easily checked, by definition, that the left hand side of Equation (43) is exactly the same as above. □

Proof of Theorem 6.9. It follows from a straightfoward computation, using Equations (39) and (40), that

$$\begin{aligned}
\{F_X, F_Y\}(\mathfrak{X}, \mathfrak{Y}) &= \mathfrak{X}(l_{[X,Y]}) - \mathfrak{Y}(l_{[X,Y]}) + \{l^1_X, f_{d_* Y}\}(\phi, \psi) \\
&\quad -\{l^2_X, f_{d_* Y}\}(\phi, \psi) - \{l^1_Y, f_{d_* X}\}(\phi, \psi) + \{l^2_Y, f_{d_* X}\}(\phi, \psi),
\end{aligned}$$

where the brackets on the right hand side are with respect to the product Poisson structure on $A^* \times \overline{A^*}$. The result now follows immediately from Lemma 6.10. □

Theorem 6.11. *For any* $\omega,\ \theta \in \Omega^1(P)$,

$$\{G_\omega, G_\theta\} = 0, \quad \{H_\omega, H_\theta\} = 0,$$

and for $\mathfrak{X} \in T_\phi(A^*),\ \mathfrak{Y} \in T_\psi(A^*)$,

$$\{G_\omega, H_\theta\}(\mathfrak{X}, \mathfrak{Y}) = -\pi_P(\omega, \theta)(q_*(\phi)) + \pi_P(\omega, \theta)(q_*(\psi)).$$

Proof. Clearly,

$$\begin{aligned}
\{G_\omega, G_\theta\}(\mathfrak{X}, \mathfrak{Y}) &= \{\langle a_*(\phi), \omega\rangle - \langle T(q_*)\mathfrak{Y}, \omega\rangle, \langle a_*(\phi), \theta\rangle - \langle T(q_*)\mathfrak{Y}, \theta\rangle\} \\
&= \{\langle T(q_*)\mathfrak{Y}, \omega\rangle, \langle T(q_*)\mathfrak{Y}, \theta\rangle\} \\
&= \{l_{q_*^*\omega}, l_{q_*^*\theta}\} \\
&= l_{\{q_*^*\omega, q_*^*\theta\}} \\
&= 0.
\end{aligned}$$

The bracket in the second-last line is the Lie algebroid bracket in the cotangent Lie algebroid T^*A^*. Note we are using the fact that a tangent Poisson structure is dual to the corresponding cotangent Lie algebroid.

The proof that $\{H_\omega, H_\theta\} = 0$ is similar. For the third identity, we have

$$\begin{aligned}
\{G_\omega, H_\theta\}(\mathfrak{X}, \mathfrak{Y}) &= -\{\langle a_*(\phi), \omega\rangle, \langle T(q_*)\mathfrak{X}, \theta\rangle\} - \{\langle T(q_*)\mathfrak{Y}, \omega\rangle, \langle a_*(\psi), \theta\rangle\} \\
&= \pi^{\#}_{A^*}(q_*^*\theta)(\langle a_*(\phi), \omega\rangle) + \pi^{\#}_{A^*}(q_*^*\omega)(\langle a_*(\psi), \theta\rangle) \\
&= \pi_{A^*}(q_*^*\theta, \delta l_{a_*^*\omega})(\phi) + \pi_{A^*}(q_*^*\omega, \delta l_{a_*^*\theta})(\psi) \\
&= -\langle a \circ a_*^*(\omega), \theta\rangle(q_*(\phi)) - \langle a \circ a_*^*(\theta), \omega\rangle(q_*(\psi)) \\
&= -\pi_P(\omega, \theta)(q_*(\phi)) + \pi_P(\omega, \theta)(q_*(\psi)).
\end{aligned}$$

Note that in the second equality, we have used the fact that the Poisson structure on the second factor is the tangent Poisson structure of $\overline{A^*}$. □

Theorem 6.12. *Suppose that (A, A^*) is a Lie bialgebroid. For any $X \in \Gamma(A)$ and $\omega \in \Omega^1(P)$,*

$$\{F_X, G_\omega\} = G_\tau \qquad \text{and} \qquad \{F_X, H_\omega\} = H_\tau,$$

where $\tau = \delta\langle a(X), \omega\rangle + \iota_{a(X)}\delta\omega$.

Proof. It is clear that

$$\begin{aligned}
\{F_X, G_\omega\}(\mathfrak{X}, \mathfrak{Y}) &= \{\mathfrak{X}(l_X), \langle a_*(\phi), \omega\rangle\} + \{\mathfrak{Y}(l_X), \langle T(q_*)\mathfrak{Y}, \omega\rangle\} \\
&\quad - \{f_{d_*X}(\phi, \psi), \langle T(q_*)\mathfrak{Y}, \omega\rangle\} \\
&= [\pi^{\#}_{A^*}(\delta l_X)](\langle a_*(\phi), \omega\rangle)(\phi) + \{l_{\delta l_X}, l_{q_*^*\omega}\}(\mathfrak{Y}) \\
&\quad - (\pi^{\#}_{A^*}(q_*^*\omega))(f_{d_*X}(\phi, \psi))
\end{aligned}$$

According to Corollary 3.9, the first term is easily seen to be

$$\begin{aligned}
[X, a_*^*\omega](\phi) &= (L_{a^*\omega}X)(\phi) + (a_*^*(\iota_{a(X)}\delta\omega))(\phi) \\
&= d_*(\iota_{a^*\omega}X)(\phi) + (\iota_{a^*\omega}d_*X)(\phi) + (a_*^*(\iota_{a(X)}\delta\omega))(\phi) \\
&= \langle a_*(\phi), \tau\rangle + (d_*X)((a^*\omega) \wedge \phi).
\end{aligned}$$

Using the standard formula for the bracket of 1-forms on a Poisson manifold (compare (19)), the second term becomes (note that the minus signs arise from the opposite Poisson structure on the second factor)

$$\begin{aligned}
\mathfrak{Y} \lrcorner \{\delta l_X, q_*^*\omega\} &= -\mathfrak{Y}(\pi_{A^*}(\delta l_X, q_*^*\omega)) - [(\pi^{\#}_{A^*}\delta l_X) \lrcorner q_*^*\delta\omega](\mathfrak{Y}) \\
&= -\langle T(q_*)\mathfrak{Y}, \delta\langle a(X), \omega\rangle\rangle - (\delta\omega)(T(q_*)(\pi^{\#}_{A^*}\delta l_X), T(q_*)\mathfrak{Y}) \\
&= -\langle T(q_*)\mathfrak{Y}, \delta\langle a(X), \omega\rangle\rangle - (\delta\omega)(a(X), T(q_*)\mathfrak{Y}) \\
&= -\langle T(q_*)\mathfrak{Y}, \delta\langle a(X), \omega\rangle\rangle - \langle T(q_*)\mathfrak{Y}, \iota_{a(X)}\delta\omega\rangle \\
&= -\langle T(q_*)\mathfrak{Y}, \tau\rangle,
\end{aligned}$$

where the third equality uses the identity $T(q_*)(\pi^{\#}_{A^*}\delta l_X) = a(X)$, which follows from Proposition 5.7.

Finally, $\pi^{\#}_{A^*}(q_*^*\omega)$ is the core element $-\overline{a^*(\omega)}$, again by 5.7, and the third term is therefore $(d_*X)(\phi \wedge (a^*\omega))$. The first equation now follows immediately. The second can be checked similarly. □

Proof of Theorem 6.2. Suppose that (Π, a_*) is a Lie algebroid morphism. Then $\mathcal{C}$ is a coisotropic submanifold in $TA^* \times T\overline{A^*}$, by Proposition 6.1, and so Theorem 6.9 implies that $(d_*[X, Y] - L_X d_*Y + L_Y d_*X)(\phi \wedge \psi) = 0$, for all $(\mathfrak{X}_\phi, \mathfrak{Y}_\psi) \in \mathcal{C}$. From Corollary 6.8 it therefore follows that

$$d_*[X, Y] = L_X d_*Y - L_Y d_*X$$

identically on $A^* \underset{P}{\times} A^*$, for all $X, Y \in \Gamma(A)$. For the other direction, we note from Proposition 6.6 that the space of functions vanishing on $\mathcal{C}$ is generated by those of the forms $F_X,\ G_\omega,\ H_\theta$ for $X \in \Gamma(A)$ and $\omega, \theta \in \Omega^1(P)$. Thus, by Theorems 6.9, 6.11 and 6.12, it is a Poisson subalgebra, and so $\mathcal{C}$ is a coisotropic submanifold. □

Example 6.13. If A is a Lie algebra $\mathfrak{g}$ such that its dual $\mathfrak{g}^*$ is also a Lie algebra, the tangent Poisson structure $T\mathfrak{g}^*$ is known to be Poisson diffeomorphic to the Lie Poisson strcture of $\mathfrak{g} \ltimes \mathfrak{g}$, the semi-direct product of the Lie algebra with itself [6]. It is easily seen that in this case the submanifold $\mathcal{C} = \{(\tau, [\tau, \theta] + \omega, \theta, \omega) | \tau,\ \theta,\ \omega \in \mathfrak{g}^*\} \subset T\mathfrak{g}^* \times T\bar{\mathfrak{g}}^*$, where $T\mathfrak{g}^*$ is identified with $\mathfrak{g}^* \times \mathfrak{g}^*$. One can check directly in fact that $\mathcal{C}$ is coisotropic if and only if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra.

7. Tangent groupoids and cotangent groupoids

In this section we present several canonical isomorphisms and some other basic results for the Lie algebroids of tangent and cotangent groupoids. These are needed in the final section and we expect them to be of value in other work also. In the construction of the Lie algebroids of Lie groupoids, we follow the conventions of [17] as modified in [18]; a different approach is given in [4].

Throughout the section we consider a fixed Lie groupoid G on base P. Then TG is a Lie groupoid on base TP with source and target $T(\alpha), T(\beta) \colon TG \longrightarrow TP$, and composition $\bullet$ defined by $\xi \bullet \eta = T(\kappa)(\xi, \eta)$ where κ is the composition in G. With this structure $(TG; G, TP; P)$ is a $\mathcal{VB}$-groupoid [21], [18]

$$\begin{array}{ccc} TG & \Longrightarrow & TP \\ {\scriptstyle p_G}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ G & \Longrightarrow & P; \end{array} \tag{44}$$

that is, each of the groupoid structure maps is a vector bundle morphism and the double source map $(p_G, T(\alpha)) \colon TG \longrightarrow G \underset{P}{\times} TP$ is a surjective submersion.

Taking the Lie algebroid of the horizontal structure in (44) gives a double vector bundle

$$\begin{array}{ccc} ATG & \xrightarrow{\ q_{TG}\ } & TP \\ {\scriptstyle A(p_G)}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ AG & \xrightarrow[\ q\ ]{} & P. \end{array} \tag{45}$$

The structure in $ATG \longrightarrow TP$ is the standard structure in the Lie algebroid of a Lie groupoid, and is denoted with the usual symbols; the zero in the fibre over $x \in TP$ is denoted $\widetilde{0}_x$. The structure in $ATG \longrightarrow AG$ is obtained by applying the Lie functor A to the operations in $TG \longrightarrow G$ and we write $\boldsymbol{+}$, $\boldsymbol{.}$ and $\boldsymbol{-}$ for the operations in $ATG \longrightarrow AG$. Thus the projection $A(p_G)$ is the result of applying the Lie functor to the morphism of groupoids $p_G \colon TG \longrightarrow G$ and the addition

$+\!\!\!+$ is similarly defined by $\mathfrak{A} +\!\!\!+ \mathfrak{B} = A(+)(\mathfrak{A}, \mathfrak{B})$. The zero above $X \in AG$ is $A(0)(X)$. It is straightforward to verify that this is a vector bundle, and that its core is $AG \longrightarrow P$.

The first result is the prototype of many similar results, and so we prove it in detail.

Theorem 7.1. *Let G be a Lie groupoid on base P. Then there is a canonical isomorphism of double vector bundles $j_G\colon TAG \longrightarrow ATG$, where ATG is as above and TAG is the tangent double vector bundle* (20) *of $AG \longrightarrow P$, which induces the identities on the side bundles AG and TP and on the cores AG. Further, j_G is an isomorphism of Lie algebroids over TP, where $TAG \longrightarrow TP$ has the tangent Lie algebroid structure of Theorem 5.1 and $ATG \longrightarrow TP$ is the Lie algebroid of $TG \rightrightarrows TP$.*

Proof. The Lie algebroid ATG is defined by a pullback diagram

$$\begin{array}{ccc} ATG & \xrightarrow{\iota_{TG}} & T^{T(\alpha)}TG \\ \big\downarrow & & \big\downarrow \\ TP & \xrightarrow[T(1)]{} & TG, \end{array}$$

where ι_{TG} is the inclusion (compare [17, III§3]), and this fits into a morphism of double vector bundles as in Figure 1. Here we are denoting the restrictions of maps by the same symbols. On the other hand, we can apply the tangent functor to the pullback diagram

$$\begin{array}{ccc} AG & \xrightarrow{\iota_{G}} & T^{\alpha}G \\ \big\downarrow & & \big\downarrow \\ P & \xrightarrow[1]{} & G, \end{array}$$

and obtain a morphism of double vector bundles as in Figure 2. From Lemma 1.5 in [18] we know that the two front faces of Figure 1 and Figure 2 are isomorphic under a restriction of the canonical involution $J\colon T^2G \longrightarrow T^2G$. Since in both Figure 1 and Figure 2 the top and bottom faces are pullbacks, it follows that J restricts to the required isomorphism $j_G\colon TAG \longrightarrow ATG$. That j_G preserves the side bundles and the core is now evident.

That $a_{TG} \circ j_G = J_P \circ T(a_G)$ follows easily by considering both a_{TG} and $T(a_G)$ as restrictions of $T^2(\beta)$ and using $J_P \circ T^2(\beta) = T^2(\beta) \circ J_G$. To show that j_G maps the bracket (27) on $TAG \longrightarrow TP$ to the bracket on $ATG \longrightarrow TP$, we need only consider sections of TAG of the form $T(X)$ and $\widehat{Y}$ for $X, Y \in \Gamma AG$. If $\overrightarrow{X}$ denotes the right-invariant vector field on G corresponding to X, and similarly for sections

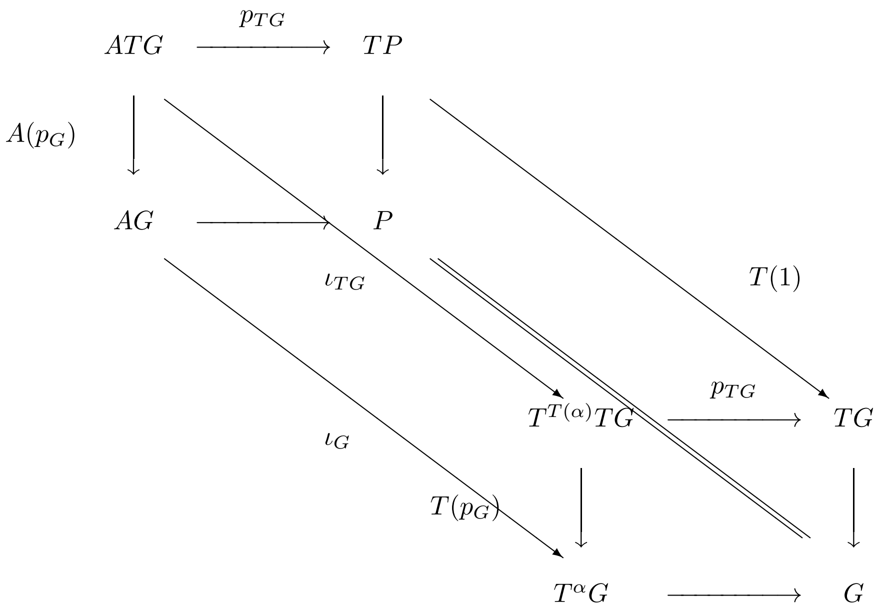

Figure 1

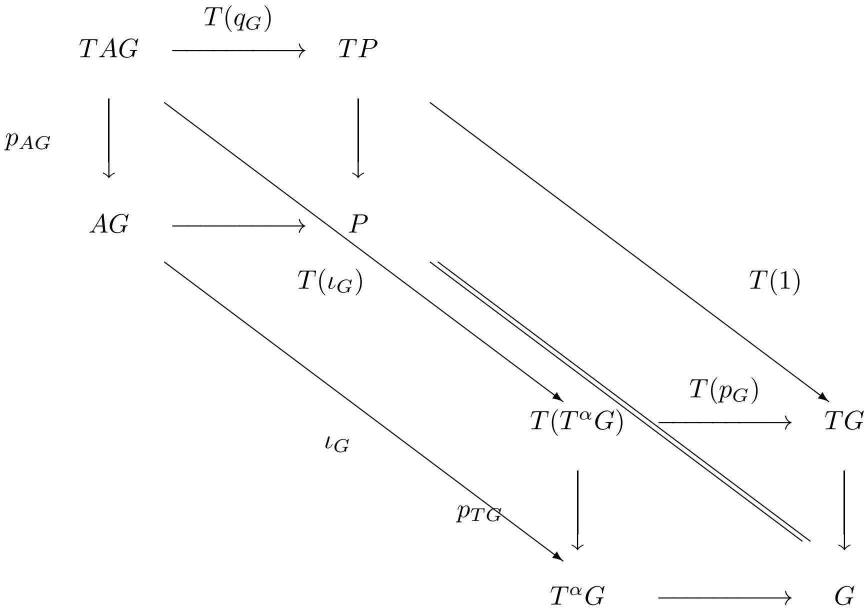

Figure 2

of ATG and right-invariant vector fields on TG, then one may see that

$$\overrightarrow{j_G \circ T(X)} = \widetilde{\overrightarrow{X}}, \qquad \overrightarrow{j_G \circ Y} = (\overrightarrow{Y})^{\breve{}},$$

for all $X, Y \in \Gamma AG$. Now the result follows easily, using Equations (27) and (26). □

Next we recall from [4], [21] the *cotangent groupoid structure* on T^*G with base the Lie algebroid dual A^*G. In the conventions we use here, the source $\widetilde{\alpha}(\omega) \in A^*_{\alpha g}G$ and target $\widetilde{\beta}(\omega) \in A^*_{\beta g}G$ of $\omega \in T^*_g G$ are given by

$$\widetilde{\alpha}(\omega)(X) = \omega(T(L_g)(X - T(1)(a(X)))), \quad \widetilde{\beta}(\omega)(Y) = \omega(T(R_g)(Y)),$$

where $X \in A_{\alpha g}G$ and $Y \in A_{\beta g}G$. Here L_g and R_g are the left and right translations in G. If $\theta \in T^*_h G$ and $\widetilde{\alpha}(\theta) = \widetilde{\beta}(\omega)$ then $\alpha h = \beta g$ and we define $\theta \bullet \omega \in T^*_{hg}G$ by

$$(\theta \bullet \omega)(Y \bullet X) = \theta(Y) + \omega(X), \qquad Y \in T_h G,\ X \in T_g G.$$

If $\phi \in A^*_m G$, then the identity element over ϕ is $\widetilde{1}_\phi \in T^*_{1_m}G$ defined by $\widetilde{1}_\phi(T(1)(x) + X) = \phi(X)$ for $X \in A_m G$, $x \in T_m(P)$. It may be verified that T^*G is a Lie groupoid on A^*G, and is a symplectic groupoid with respect to the canonical symplectic structure on T^*G. It is, further, a $\mathcal{VB}$-groupoid with respect to the usual bundle structures on $T^*G \longrightarrow G$ and $A^*G \longrightarrow P$. The core is T^*P, where $\omega \in T^*_m(P)$ corresponds to the core element $\overline{\omega} \in T^*_{1_m}G$ given by $\overline{\omega}(T(1)(x) + \overline{X}) = \omega(x + a(X))$ for $x \in T_m(P)$, $X \in A_m G$. Accordingly the Lie algebroid AT^*G has a double vector bundle structure

$$\begin{array}{ccc} AT^*G & \xrightarrow{q_{T^*G}} & A^*G \\ {\scriptstyle A(c_G)}\big\downarrow & & \big\downarrow{\scriptstyle q_*} \\ AG & \xrightarrow[q]{} & P, \end{array}$$

where the bundle structure in $AT^*G \longrightarrow AG$ arises from application of the Lie functor to $T^*G \longrightarrow G$. In particular the anchor $\widetilde{a}_*\colon AT^*G \longrightarrow TA^*G$ is a morphism of double vector bundles over $a\colon AG \longrightarrow TP$ and $\mathrm{id}\colon A^*G \longrightarrow A^*G$. Note that the core morphism of $\widetilde{a}_*$ is $a^*\colon T^*P \longrightarrow A^*G$.

Returning to (45), we can dualize the vertical structure (as in [21]) to obtain a double vector bundle

$$\begin{array}{ccc} A^\bullet TG & \longrightarrow & A^*G \\ {\scriptstyle A(p_G)_*}\big\downarrow & & \big\downarrow{\scriptstyle q_*} \\ AG & \xrightarrow[q]{} & P, \end{array}$$

with core T^*P.

Proposition 7.2. *There is a canonical isomorphism $i_G\colon AT^*G \longrightarrow A^\bullet TG$ of double vector bundles which preserves the side bundles and the cores.*

Proof. This is very similar to the proof of Proposition 5.3. We apply the functor A to the pairing $T^*G \underset{G}{\times} TG \longrightarrow G \times \mathbb{R}$ to obtain a pairing of AT^*G and ATG over AG. Since this is a suitable restriction of the pairing of TT^*G and T^2G over TG, it is nondegenerate. The remainder of the proof follows as before. □

Now we imitate the construction in Remark 5.4. Dualizing $j_G\colon TAG \longrightarrow ATG$ over AG and composing with i_G, we obtain an isomorphism of double vector bundles $j'_G\colon AT^*G \longrightarrow T^*AG$ preserving the side bundles and the cores.

Theorem 7.3. *Let G be a Lie groupoid over P. Then the isomorphism of Lie algebroids $s\colon T^*A^*G \longrightarrow AT^*G$ induced as in* [4] *by the symplectic groupoid structure on $T^*G \rightrightarrows A^*G$ is equal to $(j'_G)^{-1} \circ R$ where $R\colon T^*A^*G \longrightarrow T^*AG$ is the isomorphism of Theorem 5.5.*

The isomorphism s is induced by the Poisson bundle map $\pi^\#\colon T^*T^*G \longrightarrow TT^*G$ corresponding to the symplectic structure $\delta\Theta$ on T^*G, where Θ is the canonical 1-form, according to $\iota_{T^*G} \circ s = \pi^\# \circ \widetilde{\beta}^*$, where $\widetilde{\beta}^*\colon T^*A^*G \longrightarrow T^*T^*G$ takes $\theta \in T^*_\phi A^*G$ to its pull-back under $\widetilde{\beta}$ at the identity element $\widetilde{1}_\phi$. Thus the main work is the following result.

Lemma 7.4. *Let M be any manifold, and let T^*M have the symplectic structure $\delta\Theta$, where Θ is the canonical 1-form on T^*M. Then the corresponding Poisson bundle map $\pi^\#\colon T^*T^*M \longrightarrow TT^*M$ is equal to $(J')^{-1} \circ R'$, where $R'\colon T^*T^*M \longrightarrow T^*TM$ is the isomorphism of Theorem 5.5.*

Proof. From Proposition 5.7 it follows that $\pi^\#$ is a double vector bundle morphism preserving TM and T^*M. Take $\mathfrak{F} \in T^*T^*M$ with projections ω and X to T^*M and TM respectively. By the definition of R', it suffices to prove that

$$\langle \mathfrak{F}, \mathfrak{X}\rangle + \langle J'\pi^\#\mathfrak{F}, \xi\rangle = \langle\!\langle \mathfrak{X}, \xi\rangle\!\rangle \tag{46}$$

for all $\mathfrak{X} \in T_\omega T^*M,\ \xi \in T_X TM$ with $T(c)(\mathfrak{X}) = T(p)(\xi) = x \in TM$. It is easily seen that

$$\langle J'(\pi^\#(\mathfrak{F})), \xi\rangle = \langle\!\langle \pi^\#(\mathfrak{F}), J(\xi)\rangle\!\rangle.$$

Using Lemma 6.3, and letting ω and X also denote sections taking the given values, we have

$$\langle\!\langle \pi^\#(\mathfrak{F}), J(\xi)\rangle\!\rangle = (\pi^\#\mathfrak{F})(l_x)(\omega) + (J(\xi))(l_\omega)(x) - X\langle \omega, x\rangle.$$

Suppose first that $\mathfrak{F} = \delta l_X$. Since the $\delta\Theta$ symplectic structure on T^*M is dual to the canonical Lie algebroid structure on TM, we have $(\pi^\#\mathfrak{F})(l_x)(\omega) = l_{[X,x]}(\omega)$. Evaluating $\langle\!\langle \mathfrak{X}, \xi\rangle\!\rangle$ in the same way, and noting $\langle \mathfrak{F}, \mathfrak{X}\rangle = \mathfrak{X}(l_X)(\omega)$, it remains to prove that

$$l_{[X,x]}(\omega) + J(\xi)(l_\omega)(x) - X\langle \omega, x\rangle = \xi(l_\omega)(X) - x\langle \omega, X\rangle.$$

From [1, p.122] we have

$$J(\xi)(l_\omega)(x) = \xi(l_\omega)(X) + (\delta\omega)(X, x),$$

and the result follows. The case where $\mathfrak{F} = c^*(\omega, \theta)$ is a pullback to ω of a 1-form θ on M is similarly verified. □

Proof of Theorem 7.3. Take $\mathfrak{F} \in T^*_\phi A^*G$ with projection $X \in AG$. As in the lemma, we must prove that

$$\langle \mathfrak{F}, \mathfrak{X} \rangle + \langle j'(s(\mathfrak{F})), \xi \rangle = \langle\!\langle \mathfrak{X}, \xi \rangle\!\rangle$$

for all $\mathfrak{X} \in T_\phi A^*G$, $\xi \in T_X AG$ with $T(q_*)(\mathfrak{X}) = T(q)(\xi) = x \in TP$. As before, we have $\langle j'(s(\mathfrak{F})), \xi \rangle = \langle\!\langle s(\mathfrak{F}), j(\xi) \rangle\!\rangle$. We now regard $s(\mathfrak{F})$ as in TT^*G with $p_{T^*G}(s(\mathfrak{F})) = \widetilde{1}_\phi$ and $T(c)(s(\mathfrak{F})) = \iota_G(X) \in TG$, where ι_G is the inclusion $AG \longrightarrow TG$. Similarly we regard $j(\xi)$ as in TTG with $p_{TG}(j(\xi)) = T(1)(x)$ and $T(p)(j(\xi)) = \iota_G(X)$. In these terms $s(\mathfrak{F}) = \pi^\#(\widetilde{\beta}^*(\mathfrak{F}))$ and $j(\xi) = J(T(\iota_G)(\xi))$. So, applying Equation (46), we have

$$\begin{aligned} \langle\!\langle s(\mathfrak{F}), j(\xi) \rangle\!\rangle &= \langle J'(\pi^\#(\widetilde{\beta}^*(\mathfrak{F}))), T(\iota_G)(\xi) \rangle \\ &= \langle\!\langle T(\widetilde{1})(\mathfrak{X}), T(\iota_G)(\xi) \rangle\!\rangle - \langle \widetilde{\beta}^*(\mathfrak{F}), T(\widetilde{1})(\mathfrak{X}) \rangle \end{aligned}$$

and it is easily verified that

$$\langle\!\langle T(\widetilde{1})(\mathfrak{X}), T(\iota_G)(\xi) \rangle\!\rangle = \langle\!\langle \mathfrak{X}, \xi \rangle\!\rangle \qquad \text{and} \qquad \langle \widetilde{\beta}^*(\mathfrak{F}), T(\widetilde{1})(\mathfrak{X}) \rangle = \langle \mathfrak{F}, \mathfrak{X} \rangle.$$

This completes the proof. □

8. Poisson groupoids

A *Poisson groupoid* [24] is a Lie groupoid $G \rightrightarrows P$ together with a Poisson structure π_G on G such that the graph $\Lambda = \{(h, g, hg) \mid \alpha h = \beta g\}$ of the groupoid multiplication is a coisotropic submanifold of $G \times G \times \overline{G}$. It was shown in [24] that the manifold of identity elements of a Poisson groupoid G is coisotropic in G, and its conormal bundle $\nu^*(P)$ thereby acquires a Lie algebroid structure. This conormal bundle may be identified with A^*G, the dual vector bundle of AG, in a standard way, and we will always take A^*G with this Lie algebroid structure.

It was shown in [18, §4] that if G is a Poisson Lie group, then the cotangent bundle T^*G is an $\mathcal{LA}$-groupoid, and in particular the Poisson bundle map $T^*G \longrightarrow TG$ is a groupoid morphism from $T^*G \rightrightarrows \mathfrak{g}^*$ to the group TG; this condition is equivalent to the twisted multiplicativity equations for the dressing transformations. The following more general result was proved by Albert and Dazord [2]. The proof we give here seems conceptually simpler.

Proposition 8.1. *Let $G \rightrightarrows P$ be a Lie groupoid with Poisson structure π, and let $AG \longrightarrow P$ be its Lie algebroid. Then G is a Poisson groupoid if and only if the map*

$$\begin{array}{ccc} T^*G & \xrightarrow{\ \pi^\#\ } & TG \\ \Big\downarrow\!\Big\downarrow & & \Big\downarrow\!\Big\downarrow \\ A^*G & \xrightarrow[\ a_*\ ]{} & TP \end{array} \tag{47}$$

induced by the Poisson tensor π is a groupoid morphism.

The following lemma is quite obvious.

Lemma 8.2. *Let G_1 and G_2 be groupoids, and Λ_1, Λ_2 their graphs of multiplication. A map $\phi : G_1 \longrightarrow G_2$ is a groupoid morphism if and only if $\phi'(\Lambda_1) \subseteq \Lambda_2$, where $\phi' = \phi \times \phi \times \phi : G_1 \times G_1 \times G_1 \longrightarrow G_2 \times G_2 \times G_2$.*

Proof of Proposition 8.1. By $\Lambda \subset G \times G \times G$ we denote the graph of multiplication of the groupoid G. It is known that the graph of multiplication of the groupoid $T^*G \rightrightarrows A^*G$ is $\bar{\nu}^*\Lambda$, which is the subset of $T^*(G \times G \times G)$ obtained from the conormal bundle $\nu^*\Lambda$ by multiplying the cotangent vectors in the last factor by -1. Also, it is clear that the graph of the tangent groupoid $TG \rightrightarrows TP$ is $T\Lambda$. Thus, according to Lemma 8.2, $\pi^{\#}$ is a groupoid morphism if and only if $(\pi^{\#} \times \pi^{\#} \times \pi^{\#})(\bar{\nu}^*\Lambda) \subseteq T\Lambda$. The latter is clearly equivalent to saying that Λ is a coisotropic submanifold of $G \times G \times \overline{G}$ (see [24]), or that G is a Poisson groupoid, by definition. □

We can now prove that Lie bialgebroids do arise as infinitesimal invariants of Poisson groupoids.

Theorem 8.3. *If $G \rightrightarrows P$ is a Poisson groupoid, then (AG, A^*G) is a Lie bialgebroid.*

Proof. Since $\pi_G^{\#} : T^*G \longrightarrow TG$ is a morphism of Lie groupoids, we can apply the Lie functor and obtain $A(\pi_G^{\#}) : AT^*G \longrightarrow ATG$, a morphism of Lie algebroids over $a_* : A^*G \longrightarrow TP$.

We next prove that

$$\begin{array}{ccc} AT^*G & \xrightarrow{\quad A(\pi_G^{\#})\quad} & ATG \\ {\scriptstyle j'_G}\big\downarrow & & \big\uparrow{\scriptstyle j_G} \\ T^*AG & \xrightarrow[\quad \pi_{AG}^{\#}\quad]{} & TAG \end{array} \tag{48}$$

commutes. Let $\lambda = J'_G \circ \iota_{T^*G} \circ (j'_G)^{-1} : T^*AG \longrightarrow T^*TG$, where $\iota_{T^*G} : AT^*G \longrightarrow TT^*G$ is the inclusion. It is clear that λ is an injective morphism of double vector bundles over $\iota_G : AG \longrightarrow TG$ and $\widetilde{1} : A^*G \longrightarrow T^*G$. Now $\pi_{TG}^{\#} \circ \lambda = T(\iota_G) \circ \pi_{AG}^{\#} : T^*AG \longrightarrow TTG$, and recalling that $\iota_{TG} \circ j_G = J_G \circ T(\iota_G)$, the commutativity follows.

From Theorem 7.3 it now follows that

$$\begin{array}{ccc} AT^*G & \xrightarrow{\quad A(\pi_G^{\#})\quad} & ATG \\ {\scriptstyle s}\big\uparrow & & \big\uparrow{\scriptstyle j_G} \\ T^*A^*G & \xrightarrow[\quad \pi_{AG}^{\#} \circ R\quad]{} & TAG \end{array}$$

commutes. We know that s is an isomorphism of Lie algebroids over A^*G, and from Theorem 7.1 we know that j_G is an isomorphism of Lie algebroids over TP.

It follows that $\pi^{\#}_{AG} \circ R$ is a morphism of Lie algebroids over $a_*: A^*G \longrightarrow TP$ and by Theorem 6.2, this completes the proof. □

Note that the commutativity of (48) shows that the construction of the Poisson structure on the Lie algebroid of a Poisson groupoid generalizes the construction of the tangent Poisson structure on the tangent bundle of a Poisson manifold.

Proposition 8.4. *Let* $\mu: G \longrightarrow G'$ *be a morphism of Poisson groupoids over* $f: P \longrightarrow P'$. *Given* $\phi' \in \Gamma A^*G'$ *denote by* $\mu^{\#}(\phi')$ *the section of* A^*G *such that* $l_{\mu^{\#}(\phi')} = l_{\phi'} \circ A(\mu)$. *Then*

(1) $[\mu^{\#}(\phi'), \mu^{\#}(\psi')] = \mu^{\#}([\phi', \psi'])$ *for all* $\phi', \psi' \in \Gamma A^*G'$;

(2) $T(f)^! \circ a_* \circ A^*(\mu)^! = f^!(a'_*)$, *where* $T(f)^!: TP \longrightarrow f^!TP'$ *is the induced map over* P, $A^*(\mu)^!: f^!A^*G' \longrightarrow A^*G$ *is the dual of the induced map* $A(\mu)^!$ *over* P, *and* $f^!(a'_*): f^!A^*G' \longrightarrow f^!TP'$ *is the pullback of* a'_*.

The proof is straightforward. One may define a *morphism of Lie bialgebroids* $(A, A^*) \longrightarrow (B, B^*)$ to be a morphism of Lie algebroids $\mu: A \longrightarrow B$ over $f: P \longrightarrow Q$ such that conditions (1) and (2) above are satisfied. In the terminology of [10], a morphism of Lie bialgebroids is a morphism of Lie algebroids whose dual is a comorphism of Lie algebroids.

It would now be possible to extend to Poisson groupoids many of the technical results known for Poisson Lie groups (for example, [16]). However, we prefer to postpone this to another occasion.

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