Q-manifolds and Mackenzie Theory

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Contents

Q-manifolds

Q-manifolds are supermanifolds endowed with a homological vector field (= self-commuting odd vector field).
Features:

- A non-linear extension of the notion of a Lie algebra (together with Poisson and Schouten manifolds)
- Effective geometric language for describing algebraic structures (e.g., strongly homotopy Lie algebras, Lie algebroids, ...)

Mackenzie theory

<u>"Mackenzie theory"</u> is for Kirill Mackenzie. It embraces the following subjects:

- Double structures: double Lie groupoids and double Lie algebroids
- Lie bialgebroids and their "Drinfeld doubles"
- Duality theory for double and multiple vector bundles

Plan

I shall give an introduction to Q-manifold theory; in particular, examples of description of algebraic structures. I shall recall the notion of Lie algebroids. After that I shall speak about double Lie algebroids (originally introduced by Mackenzie in a very different way). I shall discuss application to a “Drinfeld double" of a Lie bialgebroid and generalizations such as multiple Lie algebroids (and multiple bialgebroids).

Graded manifolds and Q-manifolds

A graded manifold is a supermanifold with a privileged class of atlases where the coordinates are assigned weights in $\mathbb{Z}$, and the coordinate transformations are polynomial in coordinates with nonzero weights respecting the total weight. It is also assumed that the coordinates with nonzero weights run over the whole $\mathbb{R}$ (no restriction on range).

No relation between weight and parity (in general).

Example: any supermanifold (all weights are zero).

Example: the total space of a vector bundle where the coordinates on the base have zero weight, the linear coordinates on fibers are assigned weight 1.

Any graded manifold having only non-negative weights decomposes into a tower of affine fibrations, the first level being a vector bundle.

Q-manifolds

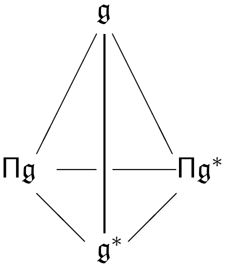
A Q-manifold is a pair (M, Q) where M is a graded manifold and $Q \in \mathfrak{X}(M)$ is an odd vector field such that $[Q, Q] = 0$ (equiv., $Q^2 = 0$). Q is called a homological vector field.
A morphism $(M_1, Q_2) \to (M_2, Q_2)$ is a smooth map $F\colon M_1 \to M_2$ such that $Q_1 \circ F^* = F^* \circ Q_2$.
Example: for an arbitrary manifold M define $\hat{M}$ so that $\Omega(M) = C^\infty(\hat{M})$. Then $(\hat{M}, d)$ is a Q-manifold. In coordinates $d = dx^a \frac{\partial}{\partial x^a}$.
Example: for a Lie algebra $\mathfrak{g}$ consider $\Pi\mathfrak{g}$ where Π is the parity reversion functor. Then $(\Pi\mathfrak{g}, Q)$ where $Q = \frac{1}{2}\xi^i\xi^j c_{ij}^k \frac{\partial}{\partial \xi^k}$, is a Q-manifold. $Q^2 = 0$ is equivalent to the Jacobi identity for c_{ij}^k.

More applications of Q-manifolds

- L_∞-algebras and L_∞-morphisms
- (Non-abelian version) A_∞-algebras
- Lie algebroids and their morphisms
- Homology of Lie algebroids
- Lie bialgebroids
- (...............)

Three manifestations of a Lie algebra

Suppose $\mathfrak{g}$ is a Lie algebra. Three other equivalent manifestations:



- Linear Poisson bracket $\{x_i, x_j\} = c_{ij}^k x_k$ on $\mathfrak{g}^*$ (Berezin–Kirillov bracket)
- Linear Schouten bracket $\{\xi_i, \xi_j\} = c_{ij}^k \xi_k$ on $\Pi\mathfrak{g}^*$
- Quadratic homological vector field $Q = \frac{1}{2}\,\xi^i \xi^j c_{ij}^k \frac{\partial}{\partial \xi^k}$ on $\Pi\mathfrak{g}$

L_∞-algebras

Consider an odd vector field $Q \in \mathfrak{X}(\mathbb{R}^{m|n})$. Let its Taylor expansion at the origin have the form

$$Q = \left(Q_0^k + \xi^i Q_i^k + \frac{1}{2}\xi^j \xi^i Q_{ij}^k + \frac{1}{3!}\xi^l \xi^j \xi^i Q_{ijl}^k + \dots \right) \frac{\partial}{\partial \xi^k}.$$

The coefficients Q_0^k, Q_i^k, Q_{ij}^k, Q_{ijl}^k, ... define a sequence of N-ary operations ($N = 0, 1, 2, 3, \dots$) on the vector space $\mathbb{R}^{n|m} = \Pi \mathbb{R}^{m|n}$, and the condition $Q^2 = 0$ expands to a linked sequence of "generalized Jacobi identities". If only the quadratic term is present, we return to the case of a Lie (super)algebra. The general case is a strong homotopy Lie algebra (L_∞-algebra)

Coordinate-free description

Given a superspace V. (For Lie algebras, $V = \mathfrak{g}$.) Each $v \in V$ defines a (constant) vector field $i_v \in V$. Define "higher derived brackets" as follows (here $N = 0, 1, 2, \ldots,$):

$$i_{\{v_1,\ldots,v_N\}_Q} := [[[\ldots[Q, v_1], v_2], \ldots, v_N](0)\,.$$

These operations are odd and symmetric (in the super sense).

Theorem

They satisfy the identities

$$\sum_{k+l=N} \sum_{(k,l)\text{-shuffles}} (-1)^\alpha \{\{v_{\sigma(1)}, \ldots, v_{\sigma(k)}\}, v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}\} = 0$$

for all $N = 0, 1, 2, \ldots$ if and only if $Q^2 = 0$. (Here $(-1)^\alpha$ is the sign prescribed by the sign rule for a permutation of homogeneous elements $v_1, \ldots, v_N \in V$.)

L_∞-morphisms

A morphism of L_∞-algebras $V_1 \to V_2$ = a morphism of the corresponding Q-manifolds (i.e., a smooth map that relates Q_1 on V_1 and Q_2 on V_2).
In coordinates: if $Q_1 = Q^k(\xi)\frac{\partial}{\partial \xi^k}$ and $Q_2 = Q^\mu(\eta)\frac{\partial}{\partial \eta^\mu}$, one has to expand

$$Q_1^i(\xi)\frac{\partial \eta^\mu}{\partial \xi^i} = Q_2^\mu(\eta(\xi))$$

into a Taylor series at the origin. (Here $F\colon (\xi^i) \mapsto (\eta^\mu(\xi))$.)

Definition of a Lie algebroid

A Lie algebroid over M is a vector bundle $E \to M$ with a Lie algebra structure on the space of sections $C^\infty(M, E)$ and a bundle map $a\colon E \to TM$ (called the anchor) satisfying

$$[u, fv] = a(u)f\, v + (-1)^{\tilde{u}\tilde{f}} f[u, v]$$

($u \in C^\infty(M, E)$ and $f \in C^\infty(M)$).
Examples: a Lie (super)algebra $\mathfrak{g}$ (here $M = \{*\}$); the tangent bundle $TM \to M$; an integrable distribution $D \subset TM$; an "action algebroid" $M \times \mathfrak{g}$.
Equivalent manifestations on "neighbors":

- Homological vector field of weight 1 on ΠE
- Poisson bracket of weight -1 on E^*
- Schouten bracket of weight -1 on ΠE^*

(structures on total spaces!).

Description via Q-manifolds

In local coordinates x^a (on the base) and ξ^i (on the fibers), we have on ΠE:

$$Q = \xi^i Q^a_i(x)\,\frac{\partial}{\partial x^a} + \frac{1}{2}\,\xi^i\xi^j Q^k_{ji}(x)\,\frac{\partial}{\partial \xi^k}\,.$$

The anchor and the Lie bracket for E are expressed by

$$a(u)f := \big[[Q, i_u)], f\big]$$

and

$$i_{[u,v]} := (-1)^{\tilde{u}}\big[[Q, i_u], i_v\big].$$

Here the map $i\colon\ C^\infty(M, E) \to \mathfrak{X}(\Pi E)$ is $u = u^i(x)e_i \mapsto i_u = (-1)^{\tilde{u}} u^i(x)\frac{\partial}{\partial \xi^i}$.

Morphisms of Lie algebroids

The definition of a morphism of Lie algebroids over different bases (due to Higgins and Mackenzie) is tricky. It is a morphism of vector bundles

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

satisfying non-obvious conditions.

Proposition (Vaintrob)

This vector bundle map is a morphism of Lie algebroids if and only if the induced map $\Phi^{\Pi}\colon \Pi E_1 \to \Pi E_2$ of the opposite vector bundles is a morphism of Q-manifolds.

Homology of Lie algebroids

For a Q-manifold M, the standard cochain complex is $(C^\infty(M), Q)$.

The standard chain complex is defined as $(\mathrm{Vol}(M), L_Q)$. Here $\mathrm{Vol}(M)$ stands for the Berezin volume forms and L_Q, for the Lie derivative w.r.t. the vector field Q. Justification: correct functorial behavior w.r.t. morphisms $F\colon M_1 \to M_2$ (the existence of forward map F_*).

Pairing of chains and cochains: $\langle f, \sigma\rangle = \int_M f\sigma$ exists always.

A "Poincaré isomorphism" $(C^\infty(M), Q) \to (\mathrm{Vol}(M), L_Q)$ exists $\Leftrightarrow$ there is an invariant non-vanishing volume form ρ $\Leftrightarrow$ the cohomology "modular class" $[\mathrm{div}_{\boldsymbol{\rho}}\, Q] \in H(C^\infty(M), Q)$ (independent of ρ) vanishes.

For Lie algebroids one obtains $(\mathrm{Vol}(\Pi E), L_Q)$ as the chain complex. (The complex appeared in Evens, Lu, and Weinstein, 1999, in a different language. Functorial property: V. Rubtsov and Th. V., in Vienna this summer.)

Definition of a Lie bialgebroid

We use the following language: a P-manifold is a Poisson manifold; an S-manifold is a Schouten manifold; a QP-manifold (a QS-manifold) possesses both Q- and P-structure (S-structure, resp.) so that the vector field is a derivation of the bracket.

Lie bialgebroids were introduced by Mackenzie and Xu; more efficient description later found by Y. Kosmann-Schwarzbach. Below is a version that uses the language of Q-manifolds.

A Lie bialgebroid over M is a Lie algebroid E over M such that E^* is also a Lie algebroid over M and so that ΠE (with the induced structure) is a QS-manifold. Equivalently: ΠE^* is a QS-manifold. (Note that there is only one type of manifestation – differently from Lie algebroids.)

Example: for $M = \{*\}$ we recover Drinfeld's Lie bialgebras.
Relevance: quantum groupoids $\Rightarrow$ Poisson groupoids $\Rightarrow$ Lie bialgebroids.

Double Lie algebroids

Double Lie algebroids were discovered by Mackenzie, who studied double Lie groupoids (in Ehresmann's sense, as groupoid objects in the category of groupoids).

$$\text{Double Lie groupoids} \Rightarrow \text{Double Lie algebroids}$$

Difficulty: no categorical definition possible; original definition is very hard. The easy part is as follows: a double Lie algebroid over M is a double vector bundle [see precise definition below]

$$\begin{array}{ccc} D & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & M \end{array}$$

such that each side (which is a vector bundle) is a Lie algebroid. The main problem is to formulate compatibility conditions.

Multiple vector bundles

A double vector bundle (a concept due to J. Pradines) over M is a fiber bundle $D \to M$ with a special structure. Trivial model: $U \times V_1 \times V_2 \times V_{12}$ where V_i, V_{ij} are vector spaces and $U \subset M$. Admissible transformations: $V_1 \times V_2 \times V_{12} \to V_1 \times V_2 \times V_{12}$ that for each V_i are linear, and for V_{12} linear in V_{12} plus an extra term bilinear in $V_1 \times V_2$. In coordinates:

$$
\begin{aligned}
u^i &= u^{i'} T_{i'}{}^{i}, \\
w^\alpha &= w^{\alpha'} T_{\alpha'}{}^{\alpha}, \\
z^\mu &= z^{\mu'} T_{\mu'}{}^{\mu} + w^{\alpha'} u^{i'} T_{i'\alpha'}{}^{\mu}\,.
\end{aligned}
$$

In particular there is a diagram as above with sides — vector bundles. Here V_1 is the standard fiber for $A \to M$; V_2, for $B \to M$; $V_1 \times V_{12}$, for $D \to B$; and $V_2 \times V_{12}$, for $D \to A$. There is also a vector bundle $K \to M$ with the standard fiber V_{12}, called the core of the double vector bundle $D \to M$.
Everything generalizes to n-fold vector bundles.

Examples

Let $E \to M$ be an ordinary vector bundle. Then there are two associated double vector bundles (very important in differential geometry and applications):

The tangent double vector bundle

$$\begin{array}{ccc} TE & \longrightarrow & E \\ {\scriptstyle Tp}\downarrow & & \downarrow{\scriptstyle p} \\ TM & \longrightarrow & M \end{array}$$

The core is isomorphic to $E \to M$.

The cotangent double vector bundle

$$\begin{array}{ccc} T^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ E^* & \longrightarrow & M \end{array}$$

The core bundle in this case is $T^*M \to M$.

Duality for multiple bundles

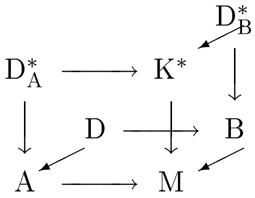
Duality theory is due to Mackenzie (and independently to Konieczna–Urbanski). Main statements:

- $D^*_A \to A$ extends to a double vector bundle over M, with the new side $K^* \to M$ and the new core $B^* \to M$
- $D^*_A \to K^*$ and $D^*_B \to K^*$ are canonically dual as vector bundles over K^* (best understood in coordinates; the duality is given by the invariant form

$$u^i u_i - w^\alpha w_\alpha$$

where the minus is absolutely essential!)

There is a 'cornering' (instead of 'pairing'):



Neighbors of a pre- double Lie algebroid

$$
\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}
\quad
\begin{array}{ccc} \Pi_A D & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}
\quad
\begin{array}{ccc} \Pi_A D^*_A & \longrightarrow & \Pi K^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}
\quad
\begin{array}{ccc} \Pi_{K^*} D^*_B & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ K^* & \longrightarrow & M \end{array}
$$

$$
\begin{array}{ccc} D^*_A & \longrightarrow & K^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}
\quad
\begin{array}{ccc} \Pi_B D & \longrightarrow & B \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array}
\quad
\begin{array}{ccc} \Pi_B D^*_B & \longrightarrow & B \\ \downarrow & & \downarrow \\ \Pi K^* & \longrightarrow & M \end{array}
\quad
\begin{array}{ccc} \Pi^2 D^*_A & \longrightarrow & \Pi K^* \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array}
$$

$$
\begin{array}{ccc} D^*_B & \longrightarrow & B \\ \downarrow & & \downarrow \\ K^* & \longrightarrow & M \end{array}
\quad
\begin{array}{ccc} \Pi^2 D & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array}
\quad
\begin{array}{ccc} \Pi_{K^*} D^*_A & \longrightarrow & K^* \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array}
\quad
\begin{array}{ccc} \Pi^2 D^*_B & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ \Pi K^* & \longrightarrow & M \end{array}
$$

Neighbors with structure on total space

Two homological fields of weights $(1,0)$ and $(0,1)$:

$$\begin{array}{ccc} \Pi^2 D & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array}$$

A Poisson or Schouten bracket of weight $(-1,-1)$ and a homological vector field of weight $(1,0)$ or $(0,1)$:

$$\begin{array}{ccc} \Pi_{K^*} D_A^* & \longrightarrow & K^* \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} \Pi_{K^*} D_B^* & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ K^* & \longrightarrow & M \end{array}$$

$$\begin{array}{ccc} \Pi^2 D_A^* & \longrightarrow & \Pi K^* \\ \downarrow & & \downarrow \\ \Pi A & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} \Pi^2 D_B^* & \longrightarrow & \Pi B \\ \downarrow & & \downarrow \\ \Pi K^* & \longrightarrow & M \end{array}$$

Main theorem

Compatibility condition for the first diagram: commutativity.
Compatibility condition for the last four diagrams: derivation property w.r.t. the bracket.

Theorem

All five conditions are equivalent. The last four conditions are the different ways of saying that $(\mathrm{D}^*_{\mathrm{A}}, \mathrm{D}^*_{\mathrm{B}})$ is a Lie bialgebroid over K^*.

Remark: that $(\mathrm{D}^*_{\mathrm{A}}, \mathrm{D}^*_{\mathrm{B}})$ is a Lie bialgebroid over K^* is the crucial part of Mackenzie's definition of a double Lie algebroid (Mackenzie's "Condition III").

A brief proof

For any Lie bialgebroid (E, E^*) the compatibility can be stated in terms of either E or E^* as a QS-structure on either ΠE or ΠE^*, resp. In our special case there is also an extra option of changing parity in the second direction. Altogether, there are four diagrams with compatibility equivalent to (D_A^*, D_B^*) being a Lie bialgebroid over K^*.

Consider one particular manifestation of this condition. The derivation property means that the flow of the vector field preserves the bracket. On the other hand, the commutativity condition for two vector fields means that the flow of one field preserves the other. Now the claim follows from functoriality: a linear transformation preserves a Lie bracket if and only if the adjoint map preserves the corresponding linear Poisson bracket and if and only if the 'Π-symmetric' map preserves the corresponding homological vector field.

Conclusion

Corollary

The double vector bundle $D \to M$ is a double Lie algebroid if and only if the homological vector fields on $\Pi^2 D \to M$ commute.

Extension to the higher case: an n-fold Lie antialgebroid is an n-fold vector bundle $E \to M$ with n commuting homological vector fields Q_i of weights δ_{ij}. Then $\Pi^n E \to M$ is, by definition, an n-fold Lie algebroid, and vice versa.

Drinfeld double of a Lie bialgebroid

According to Mackenzie: a double Lie algebroid

$$\begin{array}{ccc} T^*E = T^*E^* & \longrightarrow & E^* \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array}$$

According to Roytenberg: a Q-manifold with homological field $Q = X_{H_E} + X_{H_{E^*}}$:

$$\begin{array}{ccc} T^*\Pi E = T^*\Pi E^* & \longrightarrow & \Pi E^* \\ \downarrow & & \downarrow \\ \Pi E & \longrightarrow & M \end{array} \tag{1}$$

<u>Statement:</u> these pictures are identical up to change of parity.

More on doubles

General principle

Taking the double of an n-fold Lie bialgebroid should give an (n + 1)-fold Lie bialgebroid, with an additional property, such as a symplectic structure.

References

arXiv:math.DG/0608111

arXiv:0709.4232 [math.DG]