

Iterated bundles and Poisson geometry

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Yorkshire Differential Geometry Day,
University of York,
October 9th, 2008

1. Introduction

Poisson geometry arises in several ways:

- ▶ as a semiclassical limit of quantum theory;
- ▶ in integrable systems theory;
- ▶ as a generalization of symplectic geometry simplifying constructions such as quotients/reduction;
- ▶ as a nonlinear extension of the dual of Lie algebra theory.

It is the last aspect which concerns me today.

Given a Lie algebra \mathfrak{g} the dual has a linear Poisson structure: a linear function $\mathfrak{g}^* \rightarrow \mathbb{R}$ is an element of \mathfrak{g} and a smooth function $f: \mathfrak{g}^* \rightarrow \mathbb{R}$ can be approximated at a point φ of \mathfrak{g}^* by its derivative. Define:

$$\{f, g\}(\varphi) = \langle \varphi, [D(f)(\varphi), D(g)(\varphi)] \rangle.$$

The relationship between the bracket of vector fields on a manifold M and the canonical Poisson bracket on T^*M is similar.

2. Lie algebroids

These two constructions are examples of the Poisson structure on the dual of a Lie algebroid.

Definition: A Lie algebroid is a vector bundle $q: A \rightarrow M$ together with a bracket $[\cdot, \cdot]$ of global sections which makes ΓA an \mathbb{R} -Lie algebra, and for functions on the base obeys the Leibniz rule

$$[X, fY] = f[X, Y] + a(X)(f)Y, \quad X, Y \in \Gamma A, f: M \rightarrow \mathbb{R},$$

where $a: A \rightarrow TM$ is a vector bundle morphism, called the *anchor*. It follows that $a[X, Y] = [aX, aY]$.

Other examples: the Atiyah bundle $\frac{TP}{G}$ of a principal bundle $P(M, G)$; for a vector bundle E , all ∇_X for all connections ∇ and all vector fields X ; the cotangent bundle of an arbitrary Poisson manifold.

3. Lie algebroid duals

Define a Poisson structure on A^* . It suffices to consider pullbacks $f \circ q_*: A^* \rightarrow \mathbb{R}$ of functions f on M , and fibrewise linear functions $A^* \rightarrow \mathbb{R}$ which correspond to sections $X \in \Gamma A$. Write $l_X: A^* \rightarrow \mathbb{R}$ for the function corresponding to $X \in \Gamma A$. Define

- ▶ $\{l_X, l_Y\} = l_{[X, Y]}$;
- ▶ $\{f \circ q_*, g \circ q_*\} = 0$;
- ▶ $\{l_X, f \circ q_*\} = a(X)(f) \circ q_*$.

Applied to $A = TM$ this Poisson structure is non-degenerate and defines the canonical symplectic form on T^*M .

4. Bundles in classical mechanics

For M a smooth manifold there are four ‘second-order bundles’:

- ▶ $T^*(T^*M)$ the cotangent of the cotangent bundle;
- ▶ $T(T^*M)$;
- ▶ $T^*(TM)$;
- ▶ $T^2M = T(TM)$ the iterated tangent bundle.

The first two arise when considering the Poisson structure on T^*M : for any Poisson manifold P the ‘forms to fields’ (Hamiltonian) map is a vector bundle morphism $T^*P \rightarrow TP$. Denote $T^*(T^*M) \rightarrow T(T^*M)$ by \mathcal{H}_M for now.

Other canonical maps are the ‘Tulczyjew map’ $T(T^*M) \rightarrow T^*(TM)$ which interchanges the inner coordinates, and the canonical involution on the iterated tangent bundle, $J: T^2M \rightarrow T^2M$. In terms of ‘curves of curves’, J interchanges the order of differentiation.

5. Relationship between canonical diffeomorphisms

The diagram
$$\begin{array}{ccc} T^* T^* M & \xrightarrow{R_{TM}} & T^* TM \\ \mathcal{H}_M \downarrow & \nearrow \Theta_M & \\ TT^* M & & \end{array}$$
 commutes. (1)

◀ Return to frame 10.

Here R_{TM} is a Legendre type map. For any vector bundle $E \rightarrow M$, $R_E: T^* E^* \rightarrow T^* E$ can be written locally in terms of $E \cong M \times V$ as

$$\begin{aligned} T^* M \times V^* \times V &\rightarrow T^* M \times V \times V^*, \\ (\omega, \varphi, \mathbf{v}) &\mapsto (-\omega, \mathbf{v}, \varphi). \end{aligned}$$

The minus sign is essential. R_E is anti-symplectic.

R_E can be defined intrinsically. (KM + Ping Xu, 1994)

6. Second structures

All the vector bundles listed so far have second structures as well as the 'obvious' ones.

$$\begin{array}{ccc} TE & \xrightarrow{T(q)} & TM \\ \rho_E \downarrow & & \downarrow \rho_M \\ E & \xrightarrow{q} & M \end{array}$$

$TE \rightarrow TM$ is obtained by applying the tangent functor to all the structure maps in $E \rightarrow M$: the projection, addition, zero section, scalar multiplication.

In particular:

$$\begin{array}{ccc} T^2 M & \xrightarrow{T(\rho_M)} & TM \\ \rho_{TM} \downarrow & & \downarrow \rho_M \\ TM & \xrightarrow{\rho_M} & M \end{array}$$

$$\begin{array}{ccc} T(T^* M) & \xrightarrow{T(b_M)} & TM \\ \rho_{T^* M} \downarrow & & \downarrow \rho_M \\ T^* M & \xrightarrow{b_M} & M \end{array}$$

7. Double vector bundles

In TE the two additions satisfy an *interchange law*. Suppose given four elements, $i = 1, \dots, 4$,

$$\begin{array}{ccc}
 \xi_i & \longrightarrow & X_i \\
 \downarrow & & \downarrow \\
 e_i & \longrightarrow & m
 \end{array}
 \quad \text{of} \quad
 \begin{array}{ccc}
 TE & \xrightarrow{T(q)} & TM \\
 \rho_E \downarrow & & \rho_M \downarrow \\
 E & \xrightarrow{q} & M
 \end{array}$$

Then

$$(\xi_1 + \xi_2) \underset{TM}{+} (\xi_3 + \xi_4) = (\xi_1 \underset{TM}{+} \xi_3) + (\xi_2 \underset{TM}{+} \xi_4).$$

$+$ is the standard addition of tangent vectors; $\underset{TM}{+}$ is the addition in $TE \rightarrow TM$ (denoted $\underset{TM}{+}$ by Besse). For the sums to be defined, various conditions on the X_i, e_j are needed.

The interchange law is the main defining condition for a *double vector bundle*.

8. Duality

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

In a double vector bundle, D can be dualized in two ways.

$$\begin{array}{ccc} D^{*A} & \longrightarrow & ? \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

The bundle projection $D \rightarrow B$ is a morphism of vector bundles over $A \rightarrow M$; write K_B for its kernel. Likewise K_A for the kernel of $D \rightarrow A$. The intersection $C = K_A \cap K_B$ is the *core* of D . It is a vector bundle over M ; the two structures on D restrict to the same structure on C . The kernels are pullbacks of C , and there is a short exact sequence of bundles over A :

$$q_A^! C \longrightarrow D \longrightarrow q_A^! B$$

(Shriek denotes pullbacks.) Dualize this and we get:

$$q_A^! B^* \longrightarrow D^{*A} \longrightarrow q_A^! C^*$$

9. Duality, continued

$$\begin{array}{ccc}
 D & \longrightarrow & B \\
 \downarrow & & \downarrow \\
 A & \longrightarrow & M
 \end{array}$$

$$\text{Core} = C$$

$$\begin{array}{ccc}
 D^{*A} & \longrightarrow & C^{*} \\
 \downarrow & & \downarrow \\
 A & \longrightarrow & M
 \end{array}$$

$$\text{Core} = B^{*}$$

$$\begin{array}{ccc}
 D^{*B} & \longrightarrow & B \\
 \downarrow & & \downarrow \\
 C^{*} & \longrightarrow & M
 \end{array}$$

$$\text{Core} = A^{*}$$

Theorem: $D^{*A} \rightarrow C^{*}$ and $D^{*B} \rightarrow C^{*}$ are themselves dual.

'Proof': Take $\Phi \in D^{*A}$ and $\Psi \in D^{*B}$ projecting to same $\kappa \in C^{*}$. Say $\Phi \mapsto a \in A$ and $\Psi \mapsto b \in B$. Take any $d \in D$ which projects to a and b . The pairing is

$$\langle \Phi, \Psi \rangle_{C^{*}} = \langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B.$$

The subtraction ensures that the RHS is well-defined.

10. Duals of TE

$$\begin{array}{ccc} TE & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array}$$

Core = E

$$\begin{array}{ccc} T^*E & \longrightarrow & E^* \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array}$$

Core = T^*M

$$\begin{array}{ccc} T^*E & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E^* & \longrightarrow & M \end{array}$$

Core = E^*

The core of TE is the vertical vectors along the zero section; that is, E .

$T^*E \rightarrow TM$ denotes the dual of $TE \rightarrow TM$; it is canonically isomorphic to $T(E^*) \rightarrow TM$.

By the duality theorem, $T(E^*) \rightarrow E^*$ and $T^*E \rightarrow E^*$ are dual.

Equivalently, there is an isomorphism $T^*E^* \rightarrow T^*E$; this is R_E .

So all the maps in (1) are now defined.

11. Diagram (1) revisited

$$\begin{array}{ccc} T^*T^*M & \xrightarrow{R_{TM}} & T^*TM \\ \mathcal{H}_M \downarrow & \nearrow \Theta_M & \\ TT^*M & & \end{array}$$

- ▶ The forms to fields map \mathcal{H}_M is defined in terms of the symplectic structure on T^*M (or equivalently the bracket of vector fields);
- ▶ The Legendre map R_{TM} can be defined for any vector bundle, not just $E = TM$;
- ▶ The Tulczyjew map Θ_M is essentially a dual of the canonical involution on T^2M .

Diagram (1) consists of (\pm) symplectomorphisms between symplectic manifolds.

12. Double Lie groupoids

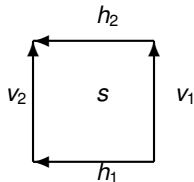
A Poisson generalization of (1) emerges from considering Lie groupoid analogues of the above constructions.

(Ordinary) Lie groupoids are the global form of Lie algebroids. Example: Given a Poisson manifold P a Lie groupoid integrating the cotangent Lie algebroid T^*P is a symplectic groupoid and provides a symplectic realization of P .

Double Lie groupoids arise in Poisson geometry as the global form of Poisson Lie groups (and Poisson Lie groupoids), and as global forms of Poisson group actions.

$$\begin{array}{ccc} S & \rightrightarrows & V \\ \Downarrow & & \Downarrow \\ H & \rightrightarrows & M \end{array}$$

S is a double Lie groupoid. The double arrows indicate groupoid structures. Picture the elements of S as squares, with horizontal edges from H and vertical edges from V . The corners are from M .



13. Lie algebroids of double Lie groupoids

The Lie algebroid of a Lie groupoid $G \rightrightarrows M$ is denoted AG . The construction is very similar to that of the Lie algebra of a Lie group. For the groupoid of all isomorphisms between the fibres of a vector bundle E , the (sections of the) Lie algebroid are all ∇_X for all connections ∇ and all X on M . For $G = M \times M$ (each arrow consists only of its endpoints), the Lie algebroid $AG = TM$.

$$\begin{array}{ccc} S & \rightrightarrows & V \\ \Downarrow & & \Downarrow \\ H & \rightrightarrows & M \end{array}$$

$$\begin{array}{ccc} A_V S & \rightrightarrows & AV \\ \downarrow & & \downarrow \\ H & \rightrightarrows & M \end{array}$$

$$\begin{array}{ccc} A_H S & \longrightarrow & V \\ \Downarrow & & \Downarrow \\ A_H & \longrightarrow & M \end{array}$$

S has two Lie groupoid structures, and we can take the Lie algebroid of either.

$A_V S$ is obtained by applying the Lie functor to the vertical groupoid structure $S \rightrightarrows H$. Because the Lie functor, like the tangent functor, preserves pullbacks, $A_V S$ is a Lie groupoid over AV . Likewise with $A_H S$.

14. Duality again

$A_V S$ is a vector bundle over H and can be dualized in the usual way. The dual has a Lie groupoid structure over $A^* K$, the Lie algebroid dual of a Lie groupoid $K \rightrightarrows M$ which arises as the core of S in a similar way to the core of a double vector bundle. Likewise with $A_H S$.

$$\begin{array}{ccccc}
 A_V S \rightrightarrows AV & & A_V^* S \rightrightarrows A^* K & & A_H^* S \longrightarrow V \\
 \downarrow & & \downarrow & & \Downarrow & \Downarrow \\
 H \rightrightarrows M & & H \rightrightarrows M & & A^* K \longrightarrow M
 \end{array}$$

Since $A_V^* S$ and $A_H^* S$ are Lie algebroid duals, they have Poisson structures, and are in fact Poisson groupoids.

Defn: A Lie groupoid $G \rightrightarrows M$ with a Poisson structure on G is a *Poisson groupoid* if the graph of the multiplication is a coisotropic submanifold of $\overline{G} \times G \times G$. (Weinstein, 1988).

15. Duality continued

For $G \rightrightarrows M$ a Poisson groupoid the base M is a coisotropic submanifold and the Lie algebroid dual A^*G (which can be regarded as the conormal bundle to the base) has a Lie algebroid structure. These make (AG, A^*G) a Lie bialgebroid.

Theorem: The Poisson groupoids $A_V^*S \rightrightarrows A^*K$ and $A_H^*S \rightrightarrows A^*K$ are dual as Poisson groupoids. ◀

Poisson groupoids are *dual* if the Lie algebroid of each is (\pm) isomorphic to the dual Lie algebroid of the other.

Thus we have $A(A_V^*S) \cong A^*(A_H^*S)$ and $A(A_H^*S) \cong A^*(A_V^*S)$ (these statements are not equivalent).

16. Canonical Poisson maps

From this structure there emerges a commutative diagram

$$\begin{array}{ccc} A^*(A_H^*S) & \xrightarrow{R} & (A(A_H S))^*{}^{AH} \\ \mathcal{H} \downarrow & \nearrow \Theta & \\ A(A_V^*S) & & \end{array}$$

which reduces to (1) in the case where $S = M^4$ consists of quadruples of points.

The maps \mathcal{H} , R , Θ are defined by lifting the definitions for manifolds to groupoids and then to double groupoids. They are (\pm) Poisson maps.

(There is a companion diagram with H and V interchanged.)

17. References

Most of the material of the talk can be found in Chapters 9 and 11 of

K. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, London Mathematical Society Lecture Note Series, no. 213, Cambridge University Press, 2005,

in the final section of

K. Mackenzie, *Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids*, math.DG/0611799

and in references there.