# Iterated bundles and Poisson geometry

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## 1. Introduction

Poisson geometry arises in several ways:

- as a semiclassical limit of quantum theory;
- in integrable systems theory;
- as a generalization of symplectic geometry simplifying constructions such as quotients/reduction;
- as a nonlinear extension of the dual of Lie algebra theory.

It is the last aspect which concerns me today.

Given a Lie algebra $\mathfrak{g}$ the dual has a linear Poisson structure: a linear function $\mathfrak{g}^* \to \mathbb{R}$ is an element of $\mathfrak{g}$ and a smooth function $f\colon \mathfrak{g}^* \to \mathbb{R}$ can be approximated at a point $\varphi$ of $\mathfrak{g}^*$ by its derivative. Define:

$$\{f, g\}(\varphi) = \langle \varphi, [D(f)(\varphi), D(g)(\varphi)] \rangle.$$

The relationship between the bracket of vector fields on a manifold $M$ and the canonical Poisson bracket on $T^*M$ is similar.

## *2. Lie algebroids*

These two constructions are examples of the Poisson structure on the dual of a Lie algebroid.

**Definition:** A *Lie algebroid* is a vector bundle $q\colon\ A \to M$ together with a bracket $[\ ,\ ]$ of global sections which makes $\Gamma A$ an $\mathbb{R}$-Lie algebra, and for functions on the base obeys the Leibniz rule

$$[X, fY] = f[X, Y] + a(X)(f)Y, \qquad X, Y \in \Gamma A,\ f\colon\ M \to \mathbb{R},$$

where $a\colon\ A \to TM$ is a vector bundle morphism, called the *anchor*. It follows that $a[X, Y] = [aX, aY]$.

Other examples: the Atiyah bundle $\frac{TP}{G}$ of a principal bundle $P(M, G)$; for a vector bundle $E$, all $\nabla_X$ for all connections $\nabla$ and all vector fields $X$; the cotangent bundle of an arbitrary Poisson manifold.

# 3. Lie algebroid duals

Define a Poisson structure on $A^*$. It suffices to consider pullbacks $f \circ q_*\colon A^* \to \mathbb{R}$ of functions $f$ on $M$, and fibrewise linear functions $A^* \to \mathbb{R}$ which correspond to sections $X \in \Gamma A$. Write $\ell_X\colon A^* \to \mathbb{R}$ for the function corresponding to $X \in \Gamma A$. Define

- $\{\ell_X, \ell_Y\} = \ell_{[X,Y]}$;
- $\{f \circ q_*, g \circ q_*\} = 0$;
- $\{\ell_X, f \circ q_*\} = a(X)(f) \circ q_*$.

Applied to $A = TM$ this Poisson structure is non-degenerate and defines the canonical symplectic form on $T^*M$.

## 4. Bundles in classical mechanics

For $M$ a smooth manifold there are four 'second-order bundles':

- $T^*(T^*M)$ the cotangent of the cotangent bundle;
- $T(T^*M)$;
- $T^*(TM)$;
- $T^2M = T(TM)$ the iterated tangent bundle.

The first two arise when considering the Poisson structure on $T^*M$: for any Poisson manifold $P$ the 'forms to fields' (Hamiltonian) map is a vector bundle morphism $T^*P \to TP$. Denote $T^*(T^*M) \to T(T^*M)$ by $\mathscr{H}_M$ for now.

Other canonical maps are the 'Tulczyjew map' $T(T^*M) \to T^*(TM)$ which interchanges the inner coordinates, and the canonical involution on the iterated tangent bundle, $J\colon\ T^2M \to T^2M$. In terms of 'curves of curves', $J$ interchanges the order of differentiation.

# 5. Relationship between canonical diffeomorphisms

The diagram
$$\begin{array}{ccc} T^*T^*M & \xrightarrow{R_{TM}} & T^*TM \\ \Big\downarrow{\scriptstyle \mathscr{H}_M} & \nearrow{\scriptstyle \Theta_M} & \\ TT^*M & & \end{array}$$
commutes. (1)

Return to frame 10.

Here $R_{TM}$ is a Legendre type map. For any vector bundle $E \to M$, $R_E\colon\ T^*E^* \to T^*E$ can be written locally in terms of $E \cong M \times V$ as

$$T^*M \times V^* \times V \to T^*M \times V \times V^*,$$
$$(\omega, \varphi, v) \mapsto (-\omega, v, \varphi).$$

The minus sign is essential. $R_E$ is anti-symplectic.

$R_E$ can be defined intrinsically. (KM + Ping Xu, 1994)

# 6. Second structures

All the vector bundles listed so far have second structures as well as the 'obvious' ones.

$$\begin{array}{ccc} TE & \xrightarrow{T(q)} & TM \\ {\scriptstyle p_E}\downarrow & & \downarrow{\scriptstyle p_M} \\ E & \xrightarrow{q} & M \end{array}$$

$TE \to TM$ is obtained by applying the tangent functor to all the structure maps in $E \to M$: the projection, addition, zero section, scalar multiplication.

In particular:

$$\begin{array}{ccc} T^2M & \xrightarrow{T(p_M)} & TM \\ {\scriptstyle p_{TM}}\downarrow & & \downarrow{\scriptstyle p_M} \\ TM & \xrightarrow{p_M} & M \end{array}$$

$$\begin{array}{ccc} T(T^*M) & \xrightarrow{T(b_M)} & TM \\ {\scriptstyle p_{T^*M}}\downarrow & & \downarrow{\scriptstyle p_M} \\ T^*M & \xrightarrow{b_M} & M \end{array}$$

## 7. Double vector bundles

In $TE$ the two additions satisfy an *interchange law*. Suppose given four elements, $i = 1, \dots, 4$,

$$\begin{array}{ccc} \xi_i & \longrightarrow & X_i \\ \big\downarrow & & \big\downarrow \\ e_i & \longrightarrow & m \end{array} \quad \text{of} \quad \begin{array}{ccc} TE & \xrightarrow{T(q)} & TM \\ {\scriptstyle p_E}\big\downarrow & & \big\downarrow{\scriptstyle p_M} \\ E & \xrightarrow{q} & M \end{array}$$

Then

$$(\xi_1 + \xi_2) \underset{TM}{+} (\xi_3 + \xi_4) = (\xi_1 \underset{TM}{+} \xi_3) + (\xi_2 \underset{TM}{+} \xi_4).$$

$+$ is the standard addition of tangent vectors; $\underset{TM}{+}$ is the addition in $TE \to TM$ (denoted $\mathbin{+\!\!\!\!\!\!+}$ by Besse). For the sums to be defined, various conditions on the $X_i, e_j$ are needed.

The interchange law is the main defining condition for a *double vector bundle*.

# 8. Duality

$$\begin{array}{ccc} D & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & M \end{array}$$

In a double vector bundle, $D$ can be dualized in two ways.

$$\begin{array}{ccc} D^{*A} & \longrightarrow & ? \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & M \end{array}$$

The bundle projection $D \to B$ is a morphism of vector bundles over $A \to M$; write $K_B$ for its kernel. Likewise $K_A$ for the kernel of $D \to A$. The intersection $C = K_A \cap K_B$ is the *core* of $D$. It is a vector bundle over $M$; the two structures on $D$ restrict to the same structure on $C$. The kernels are pullbacks of $C$, and there is a short exact sequence of bundles over $A$:

$$q_A^! C \longrightarrow D \longrightarrow q_A^! B$$

(Shriek denotes pullbacks.) Dualize this and we get:

$$q_A^! B^* \longrightarrow D^{*A} \longrightarrow q_A^! C^*$$

# 9. Duality, continued

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

Core $= C$

$$\begin{array}{ccc} D^{*A} & \longrightarrow & C^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

Core $= B^*$

$$\begin{array}{ccc} D^{*B} & \longrightarrow & B \\ \downarrow & & \downarrow \\ C^* & \longrightarrow & M \end{array}$$

Core $= A^*$

**Theorem:** $D^{*A} \to C^*$ and $D^{*B} \to C^*$ are themselves dual.

'Proof': Take $\Phi \in D^{*A}$ and $\Psi \in D^{*B}$ projecting to same $\kappa \in C^*$. Say $\Phi \mapsto a \in A$ and $\Psi \mapsto b \in B$. Take any $d \in D$ which projects to $a$ and $b$. The pairing is

$$\langle \Phi, \Psi \rangle_{C^*} = \langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B.$$

The subtraction ensures that the RHS is well-defined.

## 10. Duals of $TE$

$$\begin{array}{ccc} TE & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} T^*E & \longrightarrow & E^* \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} T^\bullet E & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E^* & \longrightarrow & M \end{array}$$

Core = $E$   Core = $T^*M$   Core = $E^*$

The core of $TE$ is the vertical vectors along the zero section; that is, $E$.

$T^\bullet E \to TM$ denotes the dual of $TE \to TM$; it is canonically isomorphic to $T(E^*) \to TM$.

By the duality theorem, $T(E^*) \to E^*$ and $T^*E \to E^*$ are dual.

Equivalently, there is an isomorphism $T^*E^* \to T^*E$; this is $R_E$.

So all the maps in (1) are now defined.

# 11. Diagram (1) revisited

$$\begin{array}{ccc} T^*T^*M & \xrightarrow{R_{TM}} & T^*TM \\ \Big\downarrow{\scriptstyle \mathscr{H}_M} & \nearrow{\scriptstyle \Theta_M} & \\ TT^*M & & \end{array}$$

- The forms to fields map $\mathscr{H}_M$ is defined in terms of the symplectic structure on $T^*M$ (or equivalently the bracket of vector fields);
- The Legendre map $R_{TM}$ can be defined for any vector bundle, not just $E = TM$;
- The Tulczyjew map $\Theta_M$ is essentially a dual of the canonical involution on $T^2M$.

Diagram (1) consists of $(\pm)$symplectomorphisms between symplectic manifolds.

## 12. Double Lie groupoids

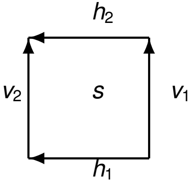
A Poisson generalization of (1) emerges from considering Lie groupoid analogues of the above constructions.

(Ordinary) Lie groupoids are the global form of Lie algebroids. Example: Given a Poisson manifold $P$ a Lie groupoid integrating the cotangent Lie algebroid $T^*P$ is a symplectic groupoid and provides a symplectic realization of $P$.

Double Lie groupoids arise in Poisson geometry as the global form of Poisson Lie groups (and Poisson Lie groupoids), and as global forms of Poisson group actions.

$$\begin{array}{ccc} S & \rightrightarrows & V \\ \downdownarrows & & \downdownarrows \\ H & \rightrightarrows & M \end{array}$$

$S$ is a double Lie groupoid. The double arrows indicate groupoid structures. Picture the elements of $S$ as squares, with horizontal edges from $H$ and vertical edges from $V$. The corners are from $M$.



# 13. Lie algebroids of double Lie groupoids

The Lie algebroid of a Lie groupoid $G \rightrightarrows M$ is denoted $AG$. The construction is very similar to that of the Lie algebra of a Lie group. For the groupoid of all isomorphisms between the fibres of a vector bundle $E$, the (sections of the) Lie algebroid are all $\nabla_X$ for all connections $\nabla$ and all $X$ on $M$. For $G = M \times M$ (each arrow consists only of its endpoints), the Lie algebroid $AG = TM$.

$$\begin{array}{ccc} S & \rightrightarrows & V \\ \downdownarrows & & \downdownarrows \\ H & \rightrightarrows & M \end{array} \qquad \begin{array}{ccc} A_V S & \rightrightarrows & AV \\ \downarrow & & \downarrow \\ H & \rightrightarrows & M \end{array} \qquad \begin{array}{ccc} A_H S & \longrightarrow & V \\ \downdownarrows & & \downdownarrows \\ AH & \longrightarrow & M \end{array}$$

$S$ has two Lie groupoid structures, and we can take the Lie algebroid of either.

$A_V S$ is obtained by applying the Lie functor to the vertical groupoid structure $S \rightrightarrows H$. Because the Lie functor, like the tangent functor, preserves pullbacks, $A_V S$ is a Lie groupoid over $AV$. Likewise with $A_H S$.

# 14. Duality again

$A_V S$ is a vector bundle over $H$ and can be dualized in the usual way. The dual has a Lie groupoid structure over $A^*K$, the Lie algebroid dual of a Lie groupoid $K \rightrightarrows M$ which arises as the core of $S$ in a similar way to the core of a double vector bundle. Likewise with $A_H S$.

$$\begin{array}{ccc} A_V S & \rightrightarrows & AV \\ \downarrow & & \downarrow \\ H & \rightrightarrows & M \end{array} \qquad \begin{array}{ccc} A_V^* S & \rightrightarrows & A^*K \\ \downarrow & & \downarrow \\ H & \rightrightarrows & M \end{array} \qquad \begin{array}{ccc} A_H^* S & \longrightarrow & V \\ \downdownarrows & & \downdownarrows \\ A^*K & \longrightarrow & M \end{array}$$

Since $A_V^* S$ and $A_H^* S$ are Lie algebroid duals, they have Poisson structures, and are in fact Poisson groupoids.

**Defn:** A Lie groupoid $G \rightrightarrows M$ with a Poisson structure on $G$ is a *Poisson groupoid* if the graph of the multiplication is a coisotropic submanifold of $\overline{G} \times G \times G$. (Weinstein, 1988).

# 15. Duality continued

For $G \rightrightarrows M$ a Poisson groupoid the base $M$ is a coisotropic submanifold and the Lie algebroid dual $A^*G$ (which can be regarded as the conormal bundle to the base) has a Lie algebroid structure. These make $(AG, A^*G)$ a Lie bialgebroid.

**Theorem:** The Poisson groupoids $A^*_V S \rightrightarrows A^*K$ and $A^*_H S \rightrightarrows A^*K$ are dual as Poisson groupoids. ◀

Poisson groupoids are *dual* if the Lie algebroid of each is $(\pm)$isomorphic to the dual Lie algebroid of the other.

Thus we have $A(A^*_V S) \cong A^*(A^*_H S)$ and $A(A^*_H S) \cong A^*(A^*_V S)$ (these statements are not equivalent).

# 16. Canonical Poisson maps

From this structure there emerges a commutative diagram

$$\begin{array}{ccc} A^*(A^*_H S) & \xrightarrow{\;R\;} & (A(A_H S))^{*AH} \\ \Big\downarrow{\scriptstyle \mathscr{H}} & \nearrow_{\Theta} & \\ A(A^*_V S) & & \end{array}$$

which reduces to (1) in the case where $S = M^4$ consists of quadruples of points.

The maps $\mathscr{H}, R, \Theta$ are defined by lifting the definitions for manifolds to groupoids and then to double groupoids. They are $(\pm)$ Poisson maps.

(There is a companion diagram with $H$ and $V$ interchanged.)

## 17. References

Most of the material of the talk can be found in Chapters 9 and 11 of

K. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids,* London Mathematical Society Lecture Note Series, no. 213, Cambridge University Press, 2005,

in the final section of

K. Mackenzie, *Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids,* math.DG/0611799

and in references there.