Remarks on Poisson actions

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Abstract

This talk sketches an overview of Poisson actions, developing my paper 'A unified approach to Poisson reduction' from 2000, and shortening some arguments in that paper.

This work was stimulated by the paper of Rui Loja Fernandes and David Iglesias Ponte (2009), 'Integrability of Poisson-Lie group actions'.

A preprint providing details of this talk will be available shortly.

1. Poisson Lie groups

Let (G, π) be a Poisson Lie group. As for any Lie group, there is the coadjoint action of G on $\mathfrak{g}^*$, which is now a Lie algebra.

And there is an infinitesimal action of $\mathfrak{g}^*$ on G. Namely, any $\xi \in \mathfrak{g}^*$ right-translates to a 1-form $\vec{\xi}$ on G and $\pi^{\#}(\vec{\xi})$ is a vector field on G. This is the dressing transformation action.

If these actions both integrate globally, then we get an action of G on G^* and an action of G^* on G. It is usual to take the action of G on G^* as a left action $(h, \varphi) \mapsto {}^h\varphi$ and the action of G^* on G as a right action, $(h, \varphi) \mapsto h^\varphi$

These actions are not by automorphisms, but each is twisted by the other action:

$$^h(\psi\varphi) = (^h\psi)(^{h^\psi}\varphi), \qquad (hg)^\psi = (h^{^g\psi})(g^\psi),$$

where $g, h \in G$ and $\varphi, \psi \in G^*$.

2. First derivation of twisted automorphism equations

These equations can be derived in an abstract setting, in two ways.

First let S be a Lie group and H and V two closed subgroups such that $S = HV = VH$,

Given $h \in H$ and $v \in V$, write $hv = ({}^h v)(h^v)$ where ${}^h v \in V$ and $h^v \in H$.

Consider $(hg)v$ where $h, g \in H$ and $v \in V$. We have

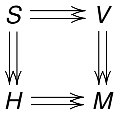
$$(hg)v = ({}^{(hg)}v)((hg)^v).$$

We also have $(hg)v = h(gv) = h(({}^g v)(g^v)) = (h({}^g v))(g^v) = {}^h({}^g v)h^{({}^g v)}(g^v)$.

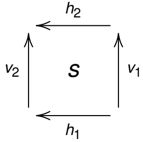
So ${}^{(hg)}v = {}^h({}^g v)$ and $(hg)^v = h^{({}^g v)}(g^v)$.

There is a similar pair of equations obtained from considering $h(vu)$.

3. Double Lie groupoids

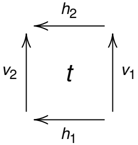


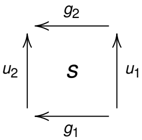
S is a double Lie groupoid. That is, S has two Lie groupoid structures, on base H and on base V, each of which is a Lie groupoid on base M, and the two structures commute in the categorical sense.



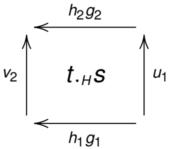
Picture the elements of S as squares s, with horizontal edges from H and vertical edges from V; the corners are from M.

h_2 is the vertical target of s; v_1 is the horizontal source, etc.



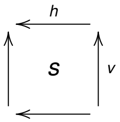


has horizontal composite (providing $v_1 = u_2$)

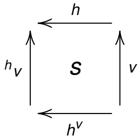


4. Vacant double Lie groupoids

S is *vacant* if (any) two touching sides determine a unique element.

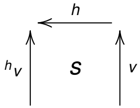


So $h \in H$ and $v \in V$ determine a unique square s and in particular determine the other two sides.



Write ${}^h v$ and h^v for the other two sides. We'll see that the horizontal target ${}^h v$ gives a left action of H on the target β_V of V, and the vertical source h^v gives a right action of V on the source α_H of H.

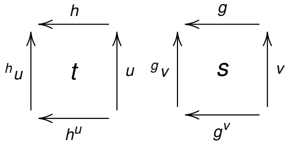
5. Note on actions



Here the source of h matches with the target of v, and ‘moves’ v to $^h v$, which has target at the target of h. Thus instead of saying that ‘H acts on V’ we say that H ‘acts on the target of V’.

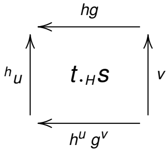
6. Second derivation of twisted automorphism equations

Compose two elements of a vacant double groupoid horizontally :



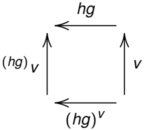
For the composition to be possible, must have $u = {}^{g}v$.

The composite must be



(by ordinary groupoid rules for the vertical edges, and by the morphism conditions for the horizontal).

But it must also be



(by the vacancy condition), so we get ${}^{(hg)}v = {}^{h}u = {}^{h}({}^{g}v)$ and $(hg)^{v} = h^{u}\, g^{v} = h^{{}^{g}v}\, g^{v}$.

7. Twisted automorphisms

From vertical compositions it follows that

$$h^{(uv)} = (h^u)^v, \qquad {}^h(vu) = ({}^h v)({}^{h^v} u),$$

(And both actions send identities to identities.)

These are the same equations as for the dressing actions of a complete Poisson Lie group, and, more generally, for a matched pair or extension bicroisée of any groups.

When (G, π) is a complete PLG, the vacant double Lie groupoid is :

$$\begin{array}{ccc} G \times G^* & \rightrightarrows & G^* \\ \downdownarrows & & \downdownarrows \\ G & \rightrightarrows & \{\cdot\} \end{array}$$

8. Actions of Poisson Lie groups I

Let σ be a Poisson Lie group action of G on a Poisson manifold P.

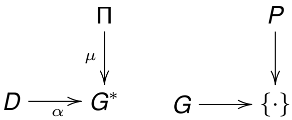
Let D be a double (symplectic) groupoid integrating G (Lu + Weinstein, 1989).

Assume that P integrates to a ssc symplectic groupoid $\Pi \rightrightarrows P$. The dual of the infinitesimal action $T^*P \to \mathfrak{g}^*$ is a Lie algebroid morphism (Xu, 1995); write $\mu\colon\ \Pi \to G^*$ for the integrated morphism of groupoids. Finally, assume that the action integrates/lifts to an action of $D \rightrightarrows G^*$ on μ.

Form the action groupid $D \lessdot \Pi \rightrightarrows \Pi$. The underlying manifold is $D \times_{G^*} \Pi$ and (d, ξ) has source ξ and target $d\xi$. Composition is $(d', \xi')(d, \xi) = (d'd, \xi)$ where ξ' must equal $d\xi$.

The underlying manifold $D \times_{G^*} \Pi$ is the pullback manifold of the source $D \to G^*$ and $\mu\colon\ \Pi \to G^*$; give it the pullback groupoid structure.

That is, both maps in the first diagram are groupoid morphisms (over the maps in the second diagram), so the pullback manifold is a subgroupoid of the Cartesian product. The base of the groupoid is $G \times P$.



9. Actions of Poisson Lie groups II

These two structures give a double Lie groupoid

$$\begin{array}{ccc} D \times_{G^*} \Pi & \rightrightarrows & \Pi \\ \downdownarrows & & \downdownarrows \\ G \lessdot P & \rightrightarrows & P \end{array}$$

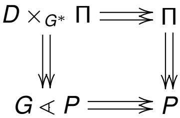
Horizontal structures are action groupoids; vertical upper structure is a pullback groupoid.

BTW: I write $G \lessdot P$ for an action groupoid, whereas most people write $G \ltimes P$.

Why? $TG = G \ltimes \mathfrak{g}$ is a standard semi-direct product group but $T^*G = G \lessdot \mathfrak{g}^*$ is an action groupoid. These constructions are different.

10. Actions of Poisson Lie groups III

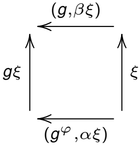
If D is vacant, then $D \times_{G^*} \Pi$ is also.



Assume that D is vacant; so $D = G \times G^*$. The action of $D \rightrightarrows G^*$ on μ simplifies to an action of G on Π.

Then $S := D \times_{G^*} \Pi = G \times \Pi = (G \lessdot P) \times_P \Pi$.

Write elements of S as (g, ξ) where $g \in G$ and $\xi \in \Pi$, and write $\varphi = \mu(\xi) \in G^*$. Then the source and targets are



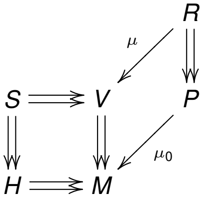
Now the twisted automorphism equations become

$$g(\xi\eta) = (g\xi)(g^{\xi}\eta), \qquad (hg)^{\xi} = (h^{g\xi})g^{\xi}$$

(Lu 1997, Fernandes + Ponte 2009).

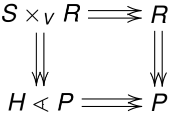
11. General construction of a double action groupoid

Consider a double Lie groupoid S (not necessarily vacant) and a morphism μ of Lie groupoids as shown.



Suppose that $S \rightrightarrows V$ acts on R and $H \rightrightarrows M$ acts on P in such a way that the action map $S \times_V R \to R$ is a groupoid morphism over the action map $H \times_M P \to P$.

Then the action groupoids form a double groupoid in which the vertical (upper) structure is the pullback groupoid.



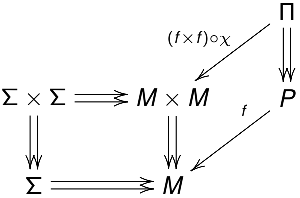
12. Actions of symplectic groupoids

This includes a construction of Xu (1992).

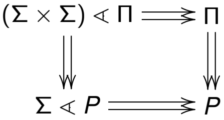
Consider a symplectic groupoid $\Sigma \rightrightarrows M$ Poisson acting on a Poisson map $f\colon\ P \to M$. Suppose that P has a ssc symplectic groupoid $\Pi \rightrightarrows P$. Xu defines two actions of Σ on Π, a left and a right action, in terms of the source and target projections of Π.

These combine to form an action of the Cartesian product groupoid $\Sigma \times \Sigma \rightrightarrows M \times M$ on the composite $\Pi \stackrel{\chi}{\to} P \times P \stackrel{f\times f}{\to} M \times M$, where χ is the anchor of Π.

This action is an action of the double Lie groupoid $\Sigma \times \Sigma$ on $\Pi \to M \times M$.



The associated double Lie groupoid is



13. Actions of Poisson Lie groups IV

Before considering Poisson groupoids, consider Poisson group actions again, but this time without assuming integrability.

For any PLG (G, π) the cotangent T^*G is a Lie groupoid with base $\mathfrak{g}^*$ and is a Lie algebroid with base G. These structures commute in the sense that the groupoid source, target, identity and multiplication are morphisms of Lie algebroids.

$$\begin{array}{ccc} T^*G & \rightrightarrows & \mathfrak{g}^* \\ \Downarrow & & \Downarrow \\ G & \rightrightarrows & \{\cdot\} \end{array}$$

In general an $\mathscr{L}\mathscr{A}$–groupoid comprises Lie groupoids $\Omega \rightrightarrows A$ and $G \rightrightarrows M$ together with Lie algebroid structures $\Omega \Longrightarrow G$ and $A \Longrightarrow M$ such that the groupoid source, target, identity and multiplication are morphisms of Lie algebroids.

$$\begin{array}{ccc} \Omega & \rightrightarrows & A \\ \Downarrow & & \Downarrow \\ G & \rightrightarrows & M \end{array}$$

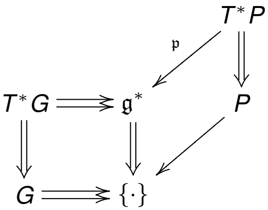
14. Actions of Poisson Lie groups V

Given a Poisson action σ of a PLG G on a Poisson manifold P, write $\mathfrak{p}\colon\ T^*P \to \mathfrak{g}^*$ for the dual of the infinitesimal of the action. This is a morphism of Lie algebroids.

Define an action of $T^*G \rightrightarrows \mathfrak{g}^*$ on $\mathfrak{p}$ by

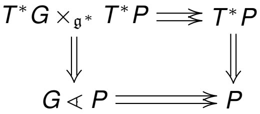
$$\theta\xi := g\xi = \xi \circ T(\sigma_{g^{-1}})$$

where $\theta \in T^*_g G$ and $\xi \in T^*P$.



Proposition: The action map $T^*G \times_{\mathfrak{g}^*} T^*P \to T^*P$ is a morphism of Lie algebroids over $\sigma\colon\ G \times P \to P$, where the domain is the pullback Lie algebroid of the source $T^*G \to \mathfrak{g}^*$ (always a surmersion) and $\mathfrak{p}\colon\ T^*P \to \mathfrak{g}^*$.

As in the groupoid case, we get an $\mathscr{L}\mathscr{A}$–groupoid



15. Vacant $\mathscr{LA}$–groupoids I

The groupoid source map $T^*G \times_{\mathfrak{g}^*} T^*P \to T^*P$ and the Lie algebroid bundle projection $T^*G \times_{\mathfrak{g}^*} T^*P \to G \lessdot P$ combine to give a diffeomorphism

$$T^*G \times_{\mathfrak{g}^*} T^*P \to (G \lessdot P) \times_P T^*P;$$

$$\begin{array}{ccc} T^*G \times_{\mathfrak{g}^*} T^*P & \rightrightarrows & T^*P \\ \Downarrow & & \Downarrow \\ G \lessdot P & \Longrightarrow & P \end{array}$$

that is, the $\mathscr{LA}$–groupoid is *vacant*.

Vacancy gives rise to an action of the Lie algebroid T^*P on $\alpha\colon\ G \lessdot P \to P$:

Given $\xi \in \Gamma T^*P$ there is a unique section ξ^α of $T^*G \times_{\mathfrak{g}^*} T^*P \Longrightarrow G \lessdot P$ which projects to ξ under the source map. The anchor of ξ^α is a vector field on $G \lessdot P$; denote it $\xi^\dagger$. There are three properties :

$$\xi^\dagger(hg) = (g\xi)^\dagger(h) \bullet \xi^\dagger(g) \qquad (1)$$

where $\bullet$ is the groupoid composition in $TG \rightrightarrows TM$.

$$T(\sigma)(\xi^\dagger(g)) = \pi_P^\#(g\xi). \qquad (2)$$

16. Vacant $\mathscr{L}\mathscr{A}$–groupoids II

For the third property, let ξ^β be the unique section of $T^*G \times_{\mathfrak{g}^*} T^*P \Longrightarrow G \lessdot P$ which projects to ξ under the target map. Then

$$[\xi^\beta, \eta^\beta] = [\xi, \eta]^\beta \tag{3}$$

for $\xi, \eta \in \Gamma T^*P$.

Aside: In the case of a PLG G the corresponding equation is equivalent to

$$\mathcal{L}_{\overleftarrow{\theta_1}}(\theta_2^{\mathrm{Ad}}) - \mathcal{L}_{\overleftarrow{\theta_2}}(\theta_1^{\mathrm{Ad}}) - [\theta_1, \theta_2]^{\mathrm{Ad}} = 0, \tag{4}$$

for $\theta_1, \theta_2 \colon G \to \mathfrak{g}^*$. Here $\theta^{\mathrm{Ad}} \colon G \to \mathfrak{g}^*$ denotes the map $g \mapsto \mathrm{Ad}^*_g(\theta(g))$ and $\overleftarrow{\theta}$ is the vector field for the left-invariant 1-form defined by θ.

Forms of equations (1) – (3) hold for any $\mathscr{L}\mathscr{A}$–groupoid.

Footnote: The isomorphism

$$T^*G \times_{\mathfrak{g}^*} T^*P \to (G \lessdot P) \times_P T^*P;$$

underlies Lu's construction of the Lie algebroid $(\mathfrak{g} \times P) \bowtie T^*P$.

17. Poisson groupoid actions I

Now consider a Poisson groupoid $G \rightrightarrows M$ acting on a Poisson map $f\colon\ P \to M$.

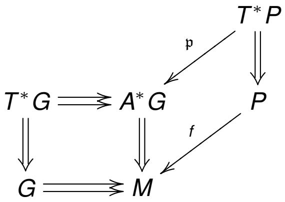
Again the dual $\mathfrak{p}\colon\ T^*P \to A^*G$ of the infinitesimal action is a morphism of Lie algebroids (He, Liu, Zhong, c.2003) and the action defines an action of $T^*G \rightrightarrows A^*G$ on $\mathfrak{p}$ as follows.

Take $\theta \in T^*_gG$ and $\xi \in T^*_uP$ with $\widetilde{\alpha}(\theta) = \mathfrak{p}(\xi)$. Any element $Z \in T_{gu}P$ can be written as $X \bullet Y$ where $X \in T_g(G)$, $Y \in T_u(P)$, and $\bullet$ is the tangent action of $TG \rightrightarrows TM$ on $TP \to TM$. (Not uniquely, of course.) Define $\theta \cdot \xi \in T^*_{gu}(P)$ by

$$\langle \theta \cdot \xi, Z\rangle = \langle \theta, X\rangle + \langle \xi, Y\rangle.$$

This is well-defined because of the $\widetilde{\alpha}(\theta) = \mathfrak{p}(\xi)$ condition.

We have a situation similar to the Poisson group case :



18. Poisson groupoid actions II

Again we get an $\mathscr{L}\mathscr{A}$–groupoid

$$\begin{array}{ccc} T^*G \times_{A^*G} T^*P & \rightrightarrows & T^*P \\ \Downarrow & & \Downarrow \\ G \lessdot P & \rightrightarrows & P \end{array}$$

Now, however, this $\mathscr{L}\mathscr{A}$–groupoid is not vacant.

The *core* of an $\mathscr{L}\mathscr{A}$–groupoid Ω is the set of elements $\kappa \in \Omega$ with groupoid source a zero and Lie algebroid bundle base point a 1. The core K is a Lie algebroid over M.

$$\begin{array}{ccc} \Omega & \rightrightarrows & A \\ \Downarrow & & \Downarrow \\ G & \rightrightarrows & M \end{array} \qquad\qquad \begin{array}{ccc} \kappa & \rightrightarrows & \overset{Y}{0} \\ \Downarrow & & \Downarrow \\ 1 & \rightrightarrows & m \end{array}$$

An $\mathscr{L}\mathscr{A}$–groupoid is vacant if and only if the core is M itself.

The core of T^*G is T^*M.

19. Poisson groupoid actions III

What is the core of $T^*G \times_{A^*G} T^*P$?

Suppose $(\varphi, \xi) \in T^*_g G \times_{A^*G} T^*_x P$ is a core element. The base point (g, x) is an identity, so $g = 1_m$ with $m = f(x)$.

Next, the source ξ of (φ, ξ) is zero, so $\varphi = \mathfrak{p}(\xi) = 0$ also. So φ is in the core of T^*G, which is T^*M.

So the core consists of all $(f^*\omega, 0_x) \in T^*_{1_m} G \times_{A^*G} T^*_x P$ for $\omega \in T^*_m M$ and $f(x) = m$. That is, it is T^*M pulled back to P over f. As a Lie algebroid it is $T^*M \lessdot f$.

So what we are looking for as a global structure is a double Lie groupoid

$$\begin{array}{ccc} D \times_{G^*} \Pi & \rightrightarrows & \Pi \\ \downdownarrows & & \downdownarrows \\ G & \rightrightarrows & P \end{array}$$

for which the core is $\Sigma \lessdot P$, where $\Sigma \rightrightarrows M$ is a symplectic groupoid for M.

20. Lie bialgebroid actions

Given a Lie bialgebroid (A, A^*) on base M and an action of A on a Poisson map $f\colon\ P \to M$, such that the dual $\mathfrak{p}\colon\ T^*P \to A^*$ of the action is a Lie algebroid morphism, there is likewise a double Lie algebroid

$$\begin{array}{ccc} T^*A \times_{A^*} T^*P & \Longrightarrow & T^*P \\ \Downarrow & & \Downarrow \\ A & \Longrightarrow & P \end{array}$$

The core here is again $T^*M \lessdot P$.

21. Conclusions

- Actions of Poisson group(oid)s can be expressed on the cotangent level and the properties of the original actions emerge naturally from the cotangent groupoids and Lie algebroids.
- Double groupoids of the type described are candidates for global forms of Poisson group(oid) actions.
- Using this viewpoint, reduction can be carried out on the cotangent level, and the symplectic groupoid level.

Note: In the references in the following frame, the two papers by myself are not on the arXiv, but pdfs are at
`http://kchmackenzie.staff.shef.ac.uk/various.html`

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