Double Lie structures and Poisson geometry

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Remarks

I want to thank most warmly the organizers and committee for creating such a vital and spirited conference.

I originally intended to give this talk in beamer but was prevented from completing the preparation at the last moment. The superb blackboards at EPFL meant that the talk I gave differed in emphasis from what would have been in the beamer presentation. I have incorporated as much as I reasonably can of the commentary into the files here.

1. Introduction

- Category theory came into being to systematize algebraic topology, but one of the simplest (nontrival) functors is differentiation — even for maps f, g between Euclidean spaces, the simplest way to express the Chain Rule is in terms of the tangent functor: $T(g \circ f) = T(g) \circ T(f)$ where $T(f)(X, Y) = (f(X), D(f)(X)(Y))$.
- The tangent functor for manifolds specializes to the Lie functor for Lie groups and Lie algebras, and to the Lie functor for Lie groupoids and Lie algebroids.
- Groups can be defined by diagrams which express the basic algebraic axioms. Requiring these diagrams to hold in the category of smooth manifolds yields the definition of Lie groups (and likewise with topological groups, etc). Poisson Lie groups cannot be defined by using the Poisson category in this way.

2. Poisson Lie groups

A Lie group G with a Poisson structure is a *Poisson Lie group*

- if the multiplication $G \times G \to G$ is a Poisson map;
- equivalently, if $\pi(gh) = T(R_h)(\pi(g)) + T(L_g)(\pi(h))$ for all $g, h \in G$;
- equivalently, if $G \to \mathfrak{g} \wedge \mathfrak{g}$, the translation to the identity of the Poisson tensor, is an adjoint– cocycle;
- equivalently, if ...

The notion of Poisson Lie group is 'not categorical':

- The inversion is anti–Poisson;
- Right and left translations are not Poisson,
  and not anti–Poisson;
- The inclusion of the identity is not a Poisson map.

3. T^*G

- Consider a Lie group G with a Poisson structure π.
- The Lie group structure gives rise to the groupoid $T^*G \rightrightarrows \mathfrak{g}^*$.
- The Poisson structure gives rise to the Lie algebroid $T^*G \to G$.

Theorem 1: If (G, π) is a Poisson Lie group then the structure maps of the cotangent groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ are Lie algebroid morphisms. ◄

$$
\begin{array}{ccc}
T^*G & \longrightarrow & G \\
\downarrow\downarrow & & \downarrow\downarrow \\
\mathfrak{g}^* & \longrightarrow & \{\cdot\}
\end{array}
$$

Source, target, inversion and inclusion of identity are Lie algebroid morphisms.

4. T^*G, p2

$$\begin{array}{ccc} T^*G & \longrightarrow & G \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathfrak{g}^* & \longrightarrow & \{\cdot\} \end{array}$$

This characterization of PLGs is ‘categorical’ in the sense that all the groupoid structure maps are Lie algebroid morphisms (no antimorphisms).

- That the source is a Lie algebroid morphism is equivalent to the bracket of left-invariant 1-forms being a left-invariant 1-form, $[\overleftarrow{\theta_1}, \overleftarrow{\theta_2}] = \overleftarrow{[\theta_1, \theta_2]_*}$;
- That the target is a Lie algebroid morphism is equivalent to the bracket of right-invariant 1-forms being a right-invariant 1-form, $[\overrightarrow{\theta_1}, \overrightarrow{\theta_2}] = \overrightarrow{[\theta_1, \theta_2]_*}$;
- That the identity inclusion $\mathfrak{g}^* \to T^*G$ is a Lie algebroid morphism is equivalent to $\mathfrak{g}^*$ being coisotropic in T^*G.

5. Inversion $T^*G \to T^*G$ is a morphism

Multiplication in T^*G is defined by

$$\langle \Phi\Psi, XY \rangle = \langle \Phi, X \rangle + \langle \Psi, Y \rangle. \qquad (1)$$

Here $\Phi \in T^*_g G$, $\Psi \in T^*_h G$ and we want $\Phi\Psi \in T^*_{gh} G$. Any element of $T_{gh}G$ can be written as a product XY with $X \in T_g G$, $Y \in T_h G$.

So $\langle \Phi^{-1}, X^{-1} \rangle = -\langle \Phi, X \rangle$. For Φ a 1-form,

$$\langle \Phi^{-1}(g), X \rangle = -\langle \Phi(g^{-1}), X^{-1} \rangle = -\langle \Phi(g^{-1}), T(i)(X) \rangle$$

where $i\colon G \to G$ is the group inversion. So $\Phi^{-1} = -i_*\Phi$.

$$\begin{array}{ccc} T^*G & \xrightarrow{i_*} & T^*G \\ \downarrow{\scriptstyle \pi^\#} & & \downarrow{\scriptstyle -\pi^\#} \\ TG & \xrightarrow{T(i)} & TG \end{array}$$

Since $i\colon G \to G$ is antiPoisson, the diagram commutes. Shifting the minus sign, we get that inversion of 1-forms preserves the anchor. It further follows that inversion preserves the bracket.

Return to frame 16.

6. T^*G*, p4*

A converse to Theorem 1 would not be useful: for example, it would require a Lie algebra structure on $\mathfrak{g}^*$ in advance. Instead:

Theorem 2: Let G be a Lie group with a Poisson structure π. Then (G, π) is a PLG if and only if the Poisson anchor $\pi^\#\colon\ T^*G \to TG$ is a Lie groupoid morphism. ◄

Here TG is the usual tangent group with multiplication $XY = T(L_g)(Y) + T(R_h)(X)$ for $X \in T_gG,\ Y \in T_hG$.

Idea of proof: That $\pi^\#$ is a groupoid morphism is equivalent to each of the groupoid structure maps commuting with the anchor. For these maps between T^*G and $\mathfrak{g}^*$, commuting with the anchors implies commuting with the brackets.

7. Poisson groupoids

Similar results hold for Poisson groupoids, and hence for symplectic groupoids.

Definition: A Lie groupoid $G \rightrightarrows P$ with a Poisson structure π on G is a *Poisson groupoid* if the graph of multiplication is a coisotropic submanifold of $\overline{G} \times G \times G$. (Weinstein, 1988) ◀

A symplectic groupoid is a Poisson groupoid for which the Poisson structure is non-degenerate.

As with PLGs, the inversion in a Poisson groupoid is antiPoisson and left and right translations are in general neither Poisson nor antiPoisson. In addition, the source and target maps have opposite polarities.

8. Poisson groupoids, p2

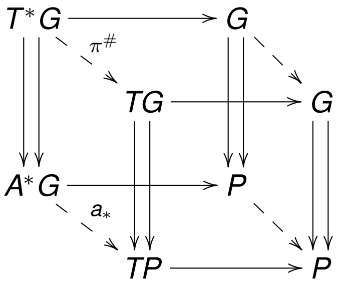
Theorem 3: A Lie groupoid $G \rightrightarrows P$ with a Poisson structure π on G is a Poisson groupoid if and only if the Poisson anchor $\pi^{\#}\colon T^*G \to TG$ is a morphism of Lie groupoids. ◀

$TG \rightrightarrows TP$ is the *tangent prolongation* Lie groupoid obtained by applying the tangent functor to all the structure maps of $G \rightrightarrows P$.

$T^*G \rightrightarrows A^*G$ is the *cotangent prolongation* Lie groupoid with base the dual of the Lie algebroid AG. For G a Lie group, $T^*G \rightrightarrows \mathfrak{g}^*$ is the action groupoid from the coadjoint action of G on $\mathfrak{g}^*$.

To require $T^*G \to TG$ to be a groupoid morphism requires a base map $A^*G \to TP$. Denote it a_*. It will be the anchor of the Lie algebroid structure on A^*G.

9. Poisson anchor



$\pi^{\#}$ is a morphism of Lie groupoids over a_* and is a morphism of Lie algebroids over G.

The basic properties of Poisson groupoids and their Lie bialgebroids can be obtained from the diagram.

Rather than do this, we will prove that for any Lie groupoid $H \rightrightarrows M$, the cotangent groupoid $T^*H \rightrightarrows A^*H$ is a Poisson (and hence symplectic) groupoid with respect to the canonical symplectic structure $d\nu$, where ν is the canonical 1-form on T^*H. This is the first real challenge that this approach faces.

10. $T^*H \rightrightarrows A^*H$

Must show that the Poisson anchor is a groupoid morphism $T^*(T^*H) \to T(T^*H)$ over a map $A^*(T^*H) \to T(A^*H)$.

Proposition: For *any manifold* H the canonical map $T^*(T^*H) \to T(T^*H)$ is the composition

$$T^*(T^*H) \stackrel{R}{\to} T^*(TH) \stackrel{\Theta^{-1}}{\to} T(T^*H).$$

(Mackenzie and Xu, '94) ◀

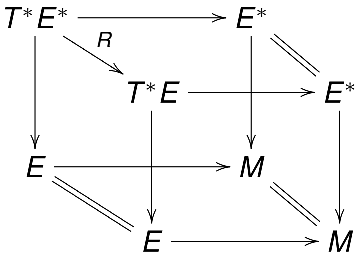
For E any vector bundle, $R_E\colon\; T^*E^* \to T^*E$ is the Legendre type map given locally by $(v, \varphi, \omega) \mapsto (\varphi, v, -\omega)$.

For H any manifold, Θ_H is the Tulczyjew map $T(T^*H) \to T^*(TH)$ which can be thought of as interchanging the inner coordinates. (See also Frame 14.)

I'll consider R and Θ individually.

11. $R\colon\ T^*E^* \to T^*E$

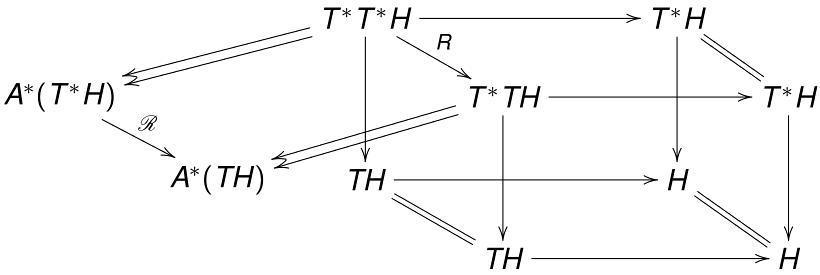
For any vector bundle $E \to M$ the map R is a morphism of double vector bundles :



It is an antisymplectomorphism for the canonical structures on the cotangents.

12. $R\colon\ T^*E^* \to T^*E$, p2

For $E = TH$ where $H \rightrightarrows M$ is a Lie groupoid, we need to show that R is a morphism of groupoids:



R will then be a morphism of three structures. The map $\mathscr{R}$ is antiPoisson.

I'll omit the proof of this and instead sketch the proof that Θ is a groupoid morphism.

13. $\Theta\colon\ T(T^*H) \to T^*(TH)$

Recall the canonical diffeomorphism $J\colon\ T^2H \to T^2H$ (for any manifold H) which interchanges the order in which derivatives are taken.

It maps the tangent bundle $T^2H = T(TH) \to TH$ of TH to the prolongation bundle obtained by applying the tangent functor to $TH \to H$.

$$\begin{array}{ccc} T^2H & \longrightarrow & T^2H \\ \Big\downarrow{\scriptstyle p_{TH}} & & \Big\downarrow{\scriptstyle T(p_H)} \\ TH & \longrightarrow & TH \end{array}$$

Θ can be thought of as the dual of J. Note:

The domain of J is the tangent bundle of TH and its dual is the cotangent bundle $T^*(TH) \to TH$.

The target of J is the prolongation of $TH \to H$ and I denote its dual by $T^\bullet(TH) \to TH$.

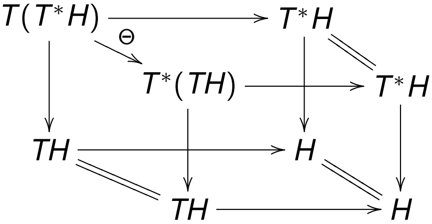
14. $\Theta\colon\ T(T^*H) \to T^*(TH)$, p2

The target of J is the prolongation of $TH \to H$ and its dual is denoted by $T^\bullet(TH) \to TH$.

The duality between TH and T^*H is defined by the pairing $T^*H \times_H TH \to \mathbb{R}$. Apply the tangent functor to this and we get a pairing $T(T^*H) \times_{TH} T(TH) \to \mathbb{R}$ which is still non-degenerate. So $T^\bullet(TH) \cong T(T^*H)$. Making this identification the dual of J is

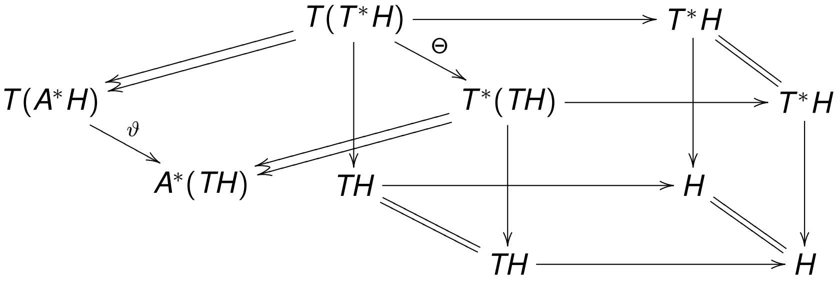
$$T(T^*H) \to T^*(TH)$$

and this is Θ. It is a morphism of double vector bundles



15. $\Theta\colon\ T(T^*H) \to T^*(TH)$, *p3*

Now suppose that $H \rightrightarrows M$ is a Lie groupoid. We will show that Θ is a morphism of groupoids :



16. $\Theta: \; T(T^*H) \to T^*(TH)$*, p4*

To show that Θ is a morphism of groupoids, it is enough to show that J is a morphism of groupoids and that the identification of $T^\bullet(TH)$ with $T(T^*H)$ is a morphism of groupoids.

For the latter, observe that the formula (1) for the multiplication in the cotangent groupoid shows that the pairing is a groupoid morphism.

That J is a morphism follows from its naturality :

The source of $T^2H \rightrightarrows T^2M$ is $T^2(\alpha)$, where α is the source of H.

By naturality, $T^2(\alpha) \circ J_H = J_M \circ T^2(\alpha)$.

Likewise with the target and the multiplication, inverse, and identity inclusion.

$$\begin{array}{ccc} T^2H & \xrightarrow{J_H} & T^2H \\ \downarrow\downarrow & & \downarrow\downarrow \\ T^2M & \xrightarrow{J_M} & T^2M \end{array}$$

17. And so . . . ?

This completes the (sketch) proof that $T^*H \rightrightarrows A^*H$, for any Lie groupoid $H \rightrightarrows M$, is a symplectic groupoid. Why prove it this way, when Coste, Dazord, Weinstein proved it in 1987 ?

My answer is that this proof extends to more general situations. The proof for $T^*H \rightrightarrows A^*H$ depends fundamentally on the relationship between the two $\mathscr{V}\mathscr{B}$-groupoids (groupoid objects in the category of vector bundles) :

$$\begin{array}{ccc} TH & \longrightarrow & H \\ \downarrow\downarrow & & \downarrow\downarrow \\ TM & \longrightarrow & M \end{array} \qquad\qquad \begin{array}{ccc} T^*H & \longrightarrow & H \\ \downarrow\downarrow & & \downarrow\downarrow \\ A^*H & \longrightarrow & M \end{array}$$

Structures of this type arise from double Lie groupoids . . .

18. Double Lie groupoids

$$\begin{array}{ccc} S & \rightrightarrows & V \\ \downdownarrows & & \downdownarrows \\ H & \rightrightarrows & M \end{array}$$

S is a double Lie groupoid. Picture the elements of S as squares, with horizontal edges from H and vertical edges from V; the corners are from M.

$$\begin{array}{ccc} A_V S & \rightrightarrows & AV \\ \downarrow & & \downarrow \\ H & \rightrightarrows & M \end{array} \qquad \begin{array}{ccc} A^*_V S & \rightrightarrows & A^*C \\ \downarrow & & \downarrow \\ H & \rightrightarrows & M \end{array}$$

$A_V S$ is obtained by applying the Lie functor to the vertical groupoid structures. $A^*_V S$ is its dual, in the sense of Pradines, and A^*C is a Lie algebroid dual which emerges from the structure of S.

$$\begin{array}{ccc} A_H S & \longrightarrow & V \\ \downdownarrows & & \downdownarrows \\ AH & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} A^*_H S & \longrightarrow & V \\ \downdownarrows & & \downdownarrows \\ A^*C & \longrightarrow & M \end{array}$$

This time we apply the Lie functor to the horizontal groupoid structures in S.

19. Final result

Consider the duals obtained on the previous slide.

$$
\begin{array}{ccc}
A^*_V S & \rightrightarrows & A^*C \\
\downarrow & & \downarrow \\
H & \rightrightarrows & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
A^*_H S & \longrightarrow & V \\
\downdownarrows & & \downdownarrows \\
A^*C & \longrightarrow & M
\end{array}
$$

Since $A^*_V S$ and $A^*_H S$ are Lie algebroid duals, they have Poisson structures.

Theorem: The groupoids $A^*_V S \rightrightarrows A^*C$ and $A^*_H S \rightrightarrows A^*C$ are Poisson groupoids, and are dual as Poisson groupoids. ◀

Poisson groupoids are *dual* if the Lie algebroid of each is isomorphic to the dual Lie algebroid of the other. (Various sign conventions exist.)

Theorem: With its canonical symplectic structure, T^*S is a symplectic double groupoid and provides a global integration of these dual Poisson groupoids. ◀

20. Concluding remarks

These results are (I believe) interesting in themselves but I also want to emphasize the value of thinking categorically – or diagrammatically – by which they were obtained.

Do these results have any connection with the work on higher category methods in TQFT and allied fields ?

It would be interesting if this is so, but my suspicion is that there is not a significant relation.

There are substantial differences between the Lie theory for multiple groupoids and current higher category methods in topology:

- The multiple groupoids studied here are strict; the methods would not readily apply to weak structures.
- Inverses are crucial to a good Lie theory — the Lie theory of semigroups with unit (a category with a single object) is a very pale shadow of the Lie theory of groups.

21. Concluding remarks, p2

When one first meets ordinary Lie groupoids, the two most basic examples are the fundamental groupoid of a manifold and the frame groupoids of structured vector bundles. Thus ordinary Lie groupoids are significant both for topology and for differential geometry.

Nonetheless, this twofold relevance seems to break down at the double level.

My own view is that the importance of the multiple Lie theory of groupoids concerns Poisson geometry and theoretical mechanics specifically and higher-order structures in differential geometry proper.

References

Most of the material of the talk can be found in Chapter 11 of

K. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids,* London Mathematical Society Lecture Note Series, no. 213, Cambridge University Press, 2005.

For the material of the final frames (17 to 19) see the final section of

K. Mackenzie, *Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids,* math.DG/0611799

and references there.