Duality for multiple vector bundles

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1. Introduction

Double vector bundles go back to the 1950s (Dombrowski) and were used in the 1960s and 1970s in some accounts of connection theory (Dieudonne, Besse) and theoretical mechanics (Tulczyjew). The first systematic account was given by Pradines (1977).

They are essentially distinct from 2-vector bundles. I'll say something about this at the end, but until then everything in this talk is for finite-dimensional smooth manifolds and all algebraic structures are strict.

The first example is TE for $E \to M$ a vector bundle. TE is a bundle over E in the usual way and is also a vector bundle over TM by applying the tangent functor to all the structure of $E \to M$. Thus the projection $TE \to TM$ is $T(q)$ where $q\colon E \to M$ is the bundle projection for E. Regard the addition in E as a map $E \times_M E \to E$. Apply T and we get $TE \times_{TM} TE \to TE$ and this defines the addition in $TE \to TM$. And so on . . .

In what follows, it is often necessary to 'apply the tangent functor' and so I always write $T(f)$ for a map f, rather than df, and so on.

2. Double vector bundles

In TE the two additions satisfy an *interchange law*. Suppose given four elements, $d_i \in TE$, $i = 1, \ldots, 4$,

$$\begin{array}{ccc} d_i & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ e_i & \longrightarrow & m \end{array} \quad \text{of} \quad \begin{array}{ccc} TE & \xrightarrow{T(q)} & TM \\ {\scriptstyle p_E}\downarrow & & \downarrow{\scriptstyle p_M} \\ E & \xrightarrow{q} & M \end{array}$$

Then

$$(d_1 + d_2) \underset{TM}{+} (d_3 + d_4) = (d_1 \underset{TM}{+} d_3) + (d_2 \underset{TM}{+} d_4).$$

Here $+$ is the standard addition of tangent vectors and $\underset{TM}{+}$ is the addition in $TE \to TM$ (denoted $\#$ by Besse). For the sums to be defined, various conditions on the X_i, e_j are needed.

The interchange law is the main defining condition for a *double vector bundle*.

3. Definition

A *double vector bundle* is a manifold D with two vector bundle structures, over bases A and B, each of which is a vector bundle on a manifold M, such that the structure maps of $D \to A$ (the bundle projection q_A, the addition $\underset{A}{+}$, the scalar multiplication, the zero section) are morphisms of vector bundles with respect to the other structure.

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

"A double vector bundle is a vector bundle object in the category of vector bundles."

The condition that the addition $\underset{A}{+}$ is a morphism with respect to the other structure is the interchange law

$$(d_1 \underset{A}{+} d_2) \underset{B}{+} (d_3 \underset{A}{+} d_4) = (d_1 \underset{B}{+} d_3) \underset{A}{+} (d_2 \underset{B}{+} d_4).$$

4. ‘Decomposed’ example

There will be more examples shortly. For now, a very simple example.

Given any three vector bundles A, B, C on the same base M, write $D = A \times_M B \times_M C$ for the fibre-product manifold.

The inverse image over $A \to M$ of the Whitney sum bundle $B \oplus C \to M$ has underlying manifold D.

Likewise, the inverse image over $B \to M$ of the Whitney sum bundle $A \oplus C \to M$ has underlying manifold D.

So D has vector bundle structures over bases A and B and is a double vector bundle.

Every double vector bundle is isomorphic to a decomposed double vector bundle (not usually in a natural way).

Note: The Whitney sum $A \oplus B \oplus C$ is a vector bundle over M. This is not part of the double vector bundle structure !

5. Duality

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \qquad\qquad \begin{array}{ccc} D^{*A} & \longrightarrow & ? \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

$D \to A$ is a vector bundle so can be dualized as usual. There is no a priori reason to expect that the result will form a double vector bundle. However . . .

Write C for the set of all elements of D which project to zero in both structures.

$$\begin{array}{ccc} d & \longrightarrow & 0^B_m \\ \downarrow & & \downarrow \\ 0^A_m & \longrightarrow & m \end{array}$$

These are closed under addition, and the two additions coincide, due to the interchange law.
So C is a vector bundle over M.
C is the *core* of D.

6. Short exact sequences

The bundle projection $D \to B$ is a morphism of vector bundles over $A \to M$. Write K_{hor} for its kernel. Every element of K_{hor} is the sum (uniquely) of a core element and a zero element in $D \to A$.

$$\begin{array}{ccc} k & \longrightarrow & 0^B_m \\ \downarrow & & \downarrow \\ a & \longrightarrow & m \end{array} \quad \text{equals} \quad \begin{array}{ccc} c & \longrightarrow & 0^B_m \\ \downarrow & & \downarrow \\ 0^A_m & \longrightarrow & m \end{array} \quad \text{plus (over } B\text{)} \quad \begin{array}{ccc} \widetilde{0}_a & \longrightarrow & 0^B_m \\ \downarrow & & \downarrow \\ a & \longrightarrow & m \end{array}$$

where $c = k -_B \widetilde{0}_a$.

The addition in K_{hor} corresponds to adding the core elements. So K_{hor} is the inverse image bundle $q_A^! C$ and we have a short exact sequence

$$0 \longrightarrow q_A^! C \longrightarrow D \longrightarrow q_A^! B \longrightarrow 0$$

(Shriek denotes inverse image.)

7. Short exact sequences, p2

The dual of the short exact sequence

$$0 \longrightarrow q_A^! C \longrightarrow D \longrightarrow q_A^! B \longrightarrow 0$$

is

$$0 \longrightarrow q_A^! B^* \longrightarrow D^{*A} \longrightarrow q_A^! C^* \longrightarrow 0$$

This suggests that there may be a double vector bundle

$$\begin{array}{ccc} D^{*A} & \longrightarrow & C^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \qquad\qquad \begin{array}{ccc} D^{*B} & \longrightarrow & B \\ \downarrow & & \downarrow \\ C^* & \longrightarrow & M \end{array}$$

and this is so. Likewise there is a double vector bundle D^{*B}.

8. Example

For $D = TE$ the core is E. Consider: the kernel of $TE \to E$ is the vectors along the zero section. And the kernel of $TE \to TM$ is the vertical vectors. Vertical vectors are tangent to the fibres and at zero can be identified with points of the fibres.

$$\begin{array}{ccc} TE & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array} \qquad\qquad \begin{array}{ccc} T^*E & \longrightarrow & E^* \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array}$$

What is the dual of TE over TM ? Apply the tangent functor to $E \times_M E^* \to \mathbb{R}$ and we get $TE \times_{TM} T(E^*) \to \mathbb{R}$, also a non-degenerate pairing. So

$$\begin{array}{ccc} TE & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array} \qquad\qquad \begin{array}{ccc} T(E^*) & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E^* & \longrightarrow & M \end{array}$$

9. The duals are dual

Theorem: $D^{*A} \to C^*$ and $D^{*B} \to C^*$ are themselves dual.

'PROOF': Take $\Phi \in D^{*A}$ and $\Psi \in D^{*B}$ projecting to same $\kappa \in C^*$. Say $\Phi \mapsto a \in A$ and $\Psi \mapsto b \in B$. Take any $d \in D$ which projects to a and b. The pairing is

$$\langle \Phi, \Psi \rangle_{C^*} = \langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B.$$

The subtraction ensures that the RHS is well-defined.

These are duals as double vector bundles. Now write X for dualization in the vertical structure and Y for dualization in the horizontal.

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} D^X & \longrightarrow & C^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} D^{XY} & \longrightarrow & C^* \\ \downarrow & & \downarrow \\ B & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} D^{XYX} & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & M \end{array}$$

The final double vector bundle is the 'flip' of the first. There is no canonical sense in which the two can be identified.

10. The duality group

Repeating from the previous slide:

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} D^X & \longrightarrow & C^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} D^{XY} & \longrightarrow & C^* \\ \downarrow & & \downarrow \\ B & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} D^{XYX} & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & M \end{array}$$

Now interchange X and Y :

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} D^Y & \longrightarrow & B \\ \downarrow & & \downarrow \\ C^* & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} D^{YX} & \longrightarrow & A \\ \downarrow & & \downarrow \\ C^* & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} D^{YXY} & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & M \end{array}$$

The results *are* canonically isomorphic. Briefly, $XYX = YXY$.

Together with $X^2 = Y^2 = I$ this shows that X, Y generate the symmetric group of order 6. Write $\mathscr{DF}_2$ for this group.

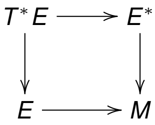
In effect $\mathscr{DF}_2$ is the symmetric group of $\{A, B, C^*\}$.

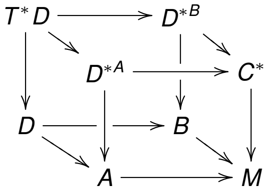
11. Triple case

Before going on to the triple case, it's reasonable to address the question: Why go further ?

$$\text{Lie algebroids} \Longrightarrow \text{Poisson structures} \Longrightarrow \text{Cotangent bundles}$$

We saw above that the cotangent of a vector bundle $E \to M$ is a double vector bundle.



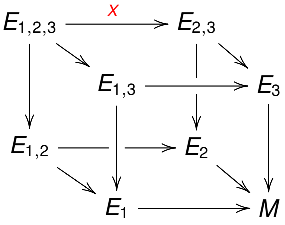


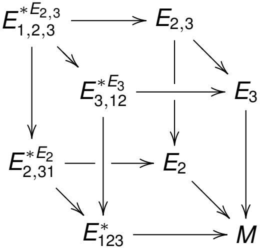
In a similar way the cotangent of a double vector bundle is a triple vector bundle. Any study of bracket structures on a double vector bundle will lead to working with triples.

And there is always curiosity. As it turns out the answer in the triple case is surprising.

12. Triple vector bundles

From here on I am describing joint work with Alfonso Gracia-Saz (LMP, 2009).

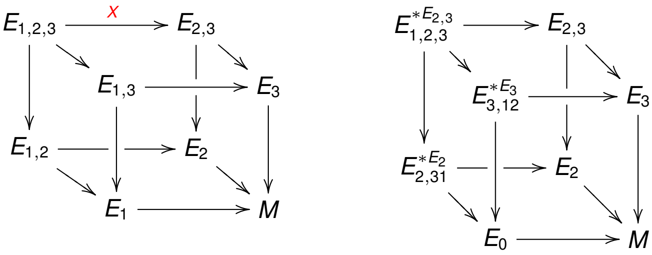




On the RHS is E^X. Imagine calculating E^{XYXZ} this way . . . it gets complicated very quickly.

E_{123} is the *ultracore*. It is the set of all elements $e \in E_{1,2,3}$ which project to zeros in $E_{1,2}$, $E_{2,3}$ and $E_{3,1}$. It is a vector bundle on base M. For brevity write $E_0 = E^*_{123}$.

13. Duality for triple vector bundles



X leaves E_2 and E_3 fixed and interchanges E_1 with E_0.

	E_1	E_2	E_3	E_0
X	E_0	E_2	E_3	E_1
Y	E_1	E_0	E_3	E_2
Z	E_1	E_2	E_0	E_3

So the group of dualization functors acts as S_4 on E_1, E_2, E_3 and E_0.

14. Duality for triple vector bundles

In effect we have a short exact sequence

$$1 \to K \to \mathscr{D}\mathscr{F}_3 \to S_4 \to 1.$$

Determine K.

S_4 is generated by $\sigma_1 = (01)$, $\sigma_2 = (02)$, $\sigma_3 = (03)$. These are subject to

$$\sigma_i^2 = 1, \qquad (\sigma_i \sigma_j)^3 = 1, \qquad (\sigma_i \sigma_j \sigma_i \sigma_k)^2 = 1,$$

i, j, k distinct.

We know that $X^2 = 1$,... and that $(XY)^3 = 1, \ldots$

Is it also true that $(XYXZ)^2 = 1$?

To settle this, look at the 'automorphisms' of E.

15. Statomorphisms

Consider first the double vector bundle case. A *statomorphism* $\varphi\colon D \to D$ is an automorphism which induces the identity on A, B and the core C. Consider the decomposed case, $D = A \times_M B \times_M C$. Then φ has the form

$$\varphi(a, b, c) = (a, b, c + \xi(a, b))$$

where $\xi\colon A \times_M B \to C$ is a bilinear map. We usually regard it as $A \otimes B \to C$.

In the triple case we have the cores E_{12}, E_{13}, E_{23} of the lower faces and the ultracore E_{123}. So an element of a decomposed triple vector bundle is $(e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123})$. Now a statomorphism is

$$\begin{aligned}\varphi(e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}) = (&e_1,\ e_2,\ e_3,\\ &e_{12} + \gamma(e_1,\ e_2),\ e_{13} + \beta(e_1,\ e_3),\ e_{23} + \alpha(e_2,\ e_3),\\ &e_{123} + \nu(e_3,\ e_{12}) + \lambda(e_1,\ e_{23}) + \mu(e_2,\ e_{13}) + \rho(e_1,\ e_2,\ e_3))\end{aligned}$$

where

$$\gamma\colon\ E_1 \otimes E_2 \to E_{12},\quad \beta\colon\ E_1 \otimes E_3 \to E_{13},\quad \alpha\colon\ E_2 \otimes E_3 \to E_{23},$$
$$\lambda\colon\ E_1 \otimes E_{23} \to E_{123},\quad \mu\colon\ E_2 \otimes E_{13} \to E_{123},\quad \nu\colon\ E_3 \otimes E_{12} \to E_{123},$$

and $\rho\colon\ E_1 \otimes E_2 \otimes E_3 \to E_{123}$ are linear maps.

16. Statomorphisms, p2

Now a dualization operator will act on statomorphisms. In the double case applying X to $\xi\colon\ A \otimes B \to C$ sends it to $-\xi\colon\ A \otimes C^* \to B^*$. We use the same letter for $A \otimes B \to C$ and the rearrangements $A \otimes C^* \to B^*$, ...

The triple case :

	γ	β	α	λ	μ	ν	ρ
X	$-\mu$	$-\nu$	α	$-\lambda$	$-\gamma$	$-\beta$	$\gamma\nu + \beta\mu - \rho$
Y	$-\lambda$	β	$-\nu$	$-\gamma$	$-\mu$	$-\alpha$	$\alpha\lambda + \gamma\nu - \rho$
Z	γ	$-\lambda$	$-\mu$	$-\beta$	$-\alpha$	$-\nu$	$\alpha\lambda + \beta\mu - \rho$

We can now calculate the effect of a word such as $(XYXZ)^2$ on the statomorphisms and we get

	γ	β	α	λ	μ	ν	ρ
X	$-\mu$	$-\nu$	α	$-\lambda$	$-\gamma$	$-\beta$	$\gamma\nu + \beta\mu - \rho$
YX	μ	α	$-\nu$	γ	λ	$-\beta$	$\rho - \beta\mu - \gamma\nu$
XYX	$-\gamma$	α	β	$-\mu$	$-\lambda$	ν	$-\rho$
$XYXZ$	$-\gamma$	μ	λ	$-\alpha$	$-\beta$	$-\nu$	$\rho - \alpha\lambda - \beta\mu$
$(XYXZ)^2$	γ	$-\beta$	$-\alpha$	$-\lambda$	$-\mu$	ν	ρ

So $(XYXZ)^2$ does not act as the identity on the statomorphisms. This certainly suggests that $(XYXZ)^2$ is a nonidentity element of the kernel. However, we have not yet made clear what the group $\mathscr{DF}_3$ is and when an element *is* the identity.

Duality of ordinary vector bundles is a contravariant functor. For double vector bundles, X and Y are contravariant functors (on suitable categories) and XY, for example, is a covariant functor.

Triple case: Consider a word W in X, Y, Z. If this is in the kernel, then W is a covariant (auto)functor on the category of triple vector bundles,

Theorem: The action of W on the group of statomorphisms is the identity if and only if W is naturally isomorphic to the identity functor through statomorphisms.

So $(XYXZ)^2 \neq 1$. Equivalently, $(XYX)Z \neq Z(XYX)$. So 'flipping' in the XY-plane does not commute with dualizing in the Z direction.

18. Sketch proof

Let $\mathtt{Cat}_i$, $i = 1, 2$, be two categories, and let $F, G\colon \mathtt{Cat}_1 \to \mathtt{Cat}_2$ be two functors. A *natural transformation* $W\colon F \to G$ is a collection of morphisms $W(E)\colon F(E) \to G(E)$ in $\mathtt{Cat}_2$ for every object E in $\mathtt{Cat}_1$ such that, given any morphism $\varphi\colon E \to E'$ in $\mathtt{Cat}_1$, we have

$$W(E') \circ F(\varphi) = G(\varphi) \circ W(E).$$

If, in addition, $W(E)$ is an isomorphism in $\mathtt{Cat}_2$ for every object E in $\mathtt{Cat}_1$, the natural transformation W is called a *natural isomorphism*.

Suppose W acts as the identity. Take a triple vector bundle E. We want to define a statomorphism $W(E)\colon E \to E^W$. Take any decomposition $P\colon E \to \overline{E}$. Applying W to P we get $P^W\colon E^W \to \overline{E}^W = \overline{E^W} = \overline{E}$. Define

$$W(E) := (P^W)^{-1} \circ P.$$

Suppose that P is replaced by some other $s \circ P$ where s is a statomorphism. Then $W(s) = s$. So $W(E)$ is well-defined. The natural isomorphism conditions are easy to verify. So is the converse.

19. Structure of the group

It is an extension of the symmetric group S_4 by the Klein four–group.

$$1 \to K_4 \to \mathscr{DF}_3 \to S_4 \to 1.$$

As an S_4–module, K_4 is isomorphic to the normal subgroup of S_4

$$\{1, (12)(30), (23)(10), (13)(20)\}$$

with action by conjugation. The group $\mathscr{DF}_3$ has order 96.

The extension is not split.

	size	ord	α	β	γ	λ	μ	ν
$(XYXZ)^2$	3	2	$-\alpha$	$-\beta$	γ	$-\lambda$	$-\mu$	ν
$(XYZ)^2$	3	4	$-\lambda$	β	ν	α	μ	$-\gamma$
$(ZYX)^2$	3	4	λ	β	$-\nu$	$-\alpha$	μ	γ
$XZXY$	6	4	λ	$-\mu$	ν	$-\alpha$	$-\beta$	$-\gamma$
XY	32	3	β	$-\nu$	λ	μ	γ	$-\alpha$
Z	12	2	$-\mu$	$-\lambda$	γ	$-\beta$	$-\alpha$	$-\nu$
$XYZYXZY$	12	4	μ	$-\lambda$	$-\gamma$	β	$-\alpha$	ν
XYZ	12	8	$-\gamma$	$-\mu$	$-\lambda$	ν	$-\beta$	α
ZYX	12	8	ν	$-\mu$	$-\alpha$	γ	$-\beta$	λ

20. Remarks

- What do the (non-identity) elements in the kernel represent? They have order 2 so are somewhat like classical duality operations. However they affect only the "internal structure". They are in some sense "invisible."
- The main consequence of the determination of $\mathscr{DF}_3$ may be expressed as: after the identity $(XY)^3 = 1$ (and its conjugates) arising from the duality of the duals of a double vector bundle, there is only one further identity, namely $(XYXZ)^4 = 1$.
- In the four–fold case we have

$$1 \to K_5 \to \mathscr{DF}_4 \to S_5 \to 1.$$

  where $K_5 = (\mathbb{Z}_2)^5$. The order is $3,840$.
- This work arose from studying bracket structures on double vector bundles.
- The groups $\mathscr{DF}_n$ are not invariants of any one n-fold vector bundle, but rather of the whole class of n-fold vector bundles, As far as we know, these groups have not arisen before.

References

See

A. Gracia-Saz and K. Mackenzie, *"Duality functors for triple vector bundles," Lett. Math. Phys.* **90**, 2009, 175 – 200.

The dual of a double object (VB-groupoid) is due to

J. Pradines, *"Remarque sur le groupoïde cotangent de Weinstein-Dazord," C. R. Acad. Sci. Paris Sér. I Math.* **306**, 1988, 557–560.

That the duals of a double vector bundle are in duality, comes from:

K. Mackenzie, *"On symplectic double groupoids and the duality of Poisson groupoids," Int. J. Math.* **10**, 1999, 435 – 456.

More detail on the double case is in Chapters 3 and 9 of:

K. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids,* London Mathematical Society Lecture Note Series, no. 213, CUP, 2005.

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